Chapter 4, part 2

EXERCISE 1. Draw the phase portrait for the equation $d\mathbf{x}/dt = A\mathbf{x}$, when the eigenvalues ($\lambda_\pm$) and eigenvectors ($\boldsymbol{\eta}_\pm$) of A are as given. The sign of $\dot{x}$ is indicated when $x = 0$ and $y > 0$.

(i) $\lambda_\pm = 1 \pm 3\mathrm{i}$ with $\boldsymbol{\eta}_\pm = (1, 2 \mp \mathrm{i})$, $\dot{x} < 0$;

(ii) $\lambda_\pm = \pm 3\mathrm{i}$ with $\boldsymbol{\eta}_\pm = (1 \pm 2\mathrm{i}, 1 \mp 3\mathrm{i})$, $\dot{x} < 0$;

(iii) $\lambda_\pm = -2 \pm \mathrm{i}$ with $\boldsymbol{\eta}_\pm = (1 \mp \mathrm{i}, 3 \pm \mathrm{i})$, $\dot{x} > 0$;

(iv) $\lambda_\pm = -1 \pm \mathrm{i}$ with $\boldsymbol{\eta}_\pm = (1, \pm\mathrm{i})$, $\dot{x} > 0$;

(v) $\lambda_\pm = 2 \pm 2\mathrm{i}$ with $\boldsymbol{\eta}_\pm = (\pm 3\mathrm{i}, 5 \mp 4\mathrm{i})$, $\dot{x} < 0$;

(vi) $\lambda_\pm = 5 \pm 3\mathrm{i}$ with $\boldsymbol{\eta}_\pm = (2 \pm 5\mathrm{i}, \mp\mathrm{i})$, $\dot{x} < 0$;

(vii) $\lambda_\pm = \pm 7\mathrm{i}$ with $\boldsymbol{\eta}_\pm = (1 \pm \mathrm{i}, -1 \pm 2\mathrm{i})$, $\dot{x} > 0$;

(viii) $\lambda_\pm = -13 \pm 17\mathrm{i}$ with $\boldsymbol{\eta}_\pm = (\pm 6\mathrm{i} - 8, 4 \mp 5\mathrm{i})$, $\dot{x} > 0$.

EXERCISE 2. Write down the general solution of the equation $d\mathbf{x}/dt = A\mathbf{x}$, when the eigenvalues ($\lambda_\pm$) and eigenvectors ($\boldsymbol{\eta}_\pm$) of A are those in the previous exercise.

EXERCISE 3. For the following equations find the eigenvalues and eigenvectors of the matrix on the right-hand side, and hence find the coordinate transformation that will put the equations into their standard simple (canonical) form. Show that this transformation has the desired effect.

(i) $\dfrac{d\mathbf{x}}{dt} = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \mathbf{x}$;

(ii) $\dfrac{d\mathbf{x}}{dt} = \begin{pmatrix} -2 & 3 \\ -6 & 4 \end{pmatrix} \mathbf{x}$;

(iii) $\dfrac{d\mathbf{x}}{dt} = \begin{pmatrix} -11 & -2 \\ 13 & -9 \end{pmatrix} \mathbf{x}$;

(iv) $\dfrac{d\mathbf{x}}{dt} = \begin{pmatrix} 7 & -5 \\ 10 & -3 \end{pmatrix} \mathbf{x}$.

EXERCISE 4. In the previous chapter we used the result that eigenvectors which correspond to distinct eigenvalues are linearly independent. Use this result to show that the real and imaginary parts of complex eigenvectors are linearly independent.

EXERCISE 5. Show how to construct a matrix with a complex conjugate pair of eigenvalues $\lambda_\pm = \rho \pm \mathrm{i}\omega$ and corresponding eigenvectors $\boldsymbol{\eta}_\pm = \mathbf{v}_1 \pm \mathrm{i}\mathbf{v}_2$. Hence find the matrices with the following eigenvalues and eigenvectors:

(i) $\lambda_\pm = 3 \pm 3\mathrm{i}$, with $\boldsymbol{\eta}_\pm = (2 \pm \mathrm{i}, 1 \mp \mathrm{i})$;

(ii) $\lambda_\pm = \pm 3\mathrm{i}$, with $\boldsymbol{\eta}_\pm = (\pm\mathrm{i}, 3 \pm 2\mathrm{i})$;

(iii) $\lambda_\pm = -2 \pm \mathrm{i}$, with $\boldsymbol{\eta}_\pm = (1 \pm \mathrm{i}, 1 \mp \mathrm{i})$.

EXERCISE 6. Find the eigenvalues and eigenvectors of the matrices occurring in the following equations, and draw the phase portrait. Find also the coordinate transformation that will put the equation into canonical form and show that this works. Write down the general solution.

(i) $\dfrac{d\mathbf{x}}{dt} = \begin{pmatrix} 5 & -4 \\ 1 & 1 \end{pmatrix} \mathbf{x}$;

(ii) $\dfrac{d\mathbf{x}}{dt} = \begin{pmatrix} -6 & 2 \\ -2 & -2 \end{pmatrix} \mathbf{x}$;

(iii) $\dfrac{d\mathbf{x}}{dt} = \begin{pmatrix} -3 & -1 \\ 1 & -5 \end{pmatrix} \mathbf{x}$;

(iv) $\dfrac{d\mathbf{x}}{dt} = \begin{pmatrix} 13 & 0 \\ 0 & 13 \end{pmatrix} \mathbf{x}$;

(v) $\dfrac{d\mathbf{x}}{dt} = \begin{pmatrix} 7 & -4 \\ 1 & 3 \end{pmatrix} \mathbf{x}$.

EXERCISE 7. Suppose that a non-zero matrix A has zero as a repeated eigenvalue, with eigenvector $\mathbf{v}$. Then we can change to coordinates referred to $\mathbf{v}_2$ and $\mathbf{v}_1 = A\mathbf{v}_2$, where $\mathbf{v}_2$ is any vector in a different direction to $\mathbf{v}$, so that $\tilde{\mathbf{x}} = \tilde{x}\mathbf{v}_1 + \tilde{y}\mathbf{v}_2$. The equation $d\mathbf{x}/dt = A\mathbf{x}$ becomes $\dfrac{d\tilde{\mathbf{x}}}{dt} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tilde{\mathbf{x}}$, hence,

$$\frac{d\tilde{x}}{dt} = \tilde{y} \quad \text{and} \quad \frac{d\tilde{y}}{dt} = 0.$$

(i) Solve the equations for $\tilde{x}(t)$ and $\tilde{y}(t)$, and hence write down the general solution for $\mathbf{x}(t)$.

(ii) Draw the phase diagram in the $(\tilde{x}, \tilde{y})$ and (x, y) planes.

(iii) Draw the phase diagram for the equation

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \mathbf{x}.$$

EXERCISE 8. Draw the phase portrait for the equation $d\mathbf{x}/dt = A\mathbf{x}$ when the eigenvalues and eigenvectors of A are as given:

(i) $\lambda_1 = 3$ with $\mathbf{v}_1 = (1, 1)$ and $\lambda_2 = -2$ with $\mathbf{v}_2 = (1, -2)$;

(ii) complex conjugate eigenvalues $\lambda_\pm = -1 \pm 3\mathrm{i}$, with $\dot{x} < 0$ when $x = 0$ and $y > 0$;

(iii) a single eigenvalue $\lambda = 13$ with eigenvector $(3, 2)$, and $\dot{x} > 0$ when $x = 0$ and $y > 0$;

(iv) $\lambda_1 = -2$ with $\mathbf{v}_1 = (2, 1)$ and $\lambda_2 = -3$ with $\mathbf{v}_2 = (1, -1)$;

(v) a single eigenvalue $\lambda = -3$ with eigenvector $(1, -1)$, and $\dot{x} > 0$ when $x = 0$ and $y > 0$;

(vi) $\lambda_\pm = \pm 2\mathrm{i}$, where $\dot{y} < 0$ when $y = 0$ and $x > 0$;

(vii) $\lambda_1 = 1$ with $\mathbf{v}_1 = (3, 2)$ and $\lambda_2 = 5$ with $\mathbf{v}_2 = (1, -4)$;

(viii) $\lambda_\pm = 5 \pm \mathrm{i}$, and $\dot{y} > 0$ when $y = 0$ and $x > 0$;

(ix) a single eigenvalue $\lambda = -7$, with the matrix A being a multiple of the identity.