Chapter 4, part 1

EXERCISE 1. By choosing an appropriate collection of new variables $x_1, \ldots, x_n$, rewrite the nth order differential equation

$$\frac{d^n x}{dt^n} = f\left(\frac{d^{n-1}x}{dt^{n-1}}, \ldots, \frac{dx}{dt}, x, t\right)$$

as a set of n coupled linear first order equations. Find the conditions on the function $f(x_n, \ldots, x_1, t)$ for the original differential equation to have a unique solution.

EXERCISE 2. Show that if a homogeneous linear system

$$\begin{cases} \dfrac{dx}{dt} = a_1(t)x + b_1(t)y, \\ \dfrac{dy}{dt} = a_2(t)x + b_2(t)y \end{cases} \tag{1}$$

has two solutions

$$\begin{cases} x = x_1(t) \\ y = y_1(t) \end{cases} \quad \text{and} \quad \begin{cases} x = x_2(t) \\ y = y_2(t) \end{cases}, \tag{2}$$

then any linear combination

$$\begin{cases} x = c_1 x_1(t) + c_2 x_2(t) \\ y = c_1 y_1(t) + c_2 y_2(t) \end{cases} \tag{3}$$

with constants $c_1, c_2 \in \mathbb{R}$ is a solution as well.

EXERCISE 3. Show that given a homogeneous linear system (1) with continuous coefficients in $[a, b]$, then the Wronskian $W(t)$ of two solutions (2) satisfies the first order ODE

$$\frac{dW}{dt} = (a_1(t) + b_2(t))\, W.$$

Deduce that this Wronskian is either identically 0 or nowhere zero in $[a, b]$.

EXERCISE 4. Show that if two solutions (2) of (1) have a Wronskian that does not vanish in $[a, b]$, then (3) gives the general solution of the system.

EXERCISE 5. Suppose that if we have an inhomogenous system

$$\begin{cases} \dfrac{dx}{dt} = a_1(t)x + b_1(t)y + f_1(t), \\ \dfrac{dy}{dt} = a_2(t)x + b_2(t)y + f_2(t) \end{cases} \tag{4}$$

and one particular solution

$$\begin{cases} x = x_p(t) \\ y = y_p(t). \end{cases} \tag{5}$$

Show that, if $x_1(t)$ and $x_2(t)$ are linearly independent solutions of the homogeneous system, then the general solution of (4) is

$$\begin{cases} x = c_1 x_1(t) + c_2 x_2(t) + x_p(t) \\ y = c_1 y_1(t) + c_2 y_2(t) + y_p(t) \end{cases}$$

EXERCISE 6. Suppose that we have a second order linear ODE $x''(t) + p(t)x'(t) + q(t)x(t) = 0$, and two solutions $x_1(t)$ and $x_2(t)$. Write it as a linear system of two first order ODEs. Check that if

$$\begin{cases} x = x_1(t) \\ y = y_1(t) \end{cases} \quad \text{and} \quad \begin{cases} x = x_2(t) \\ y = y_2(t) \end{cases}$$

are the corresponding solutions of the system, then the Wronskian as defined in this chapter coincides with the definition that we had before for second order ODEs.

EXERCISE 7. Find the general solutions of the following differential equations by converting them into a single second-order equation. Also find the solution that satisfies the given initial conditions.

(i) $\begin{cases} \dot{x} = 4x - y \\ \dot{y} = 2x + y + t^2, \end{cases}$ $x(0) = 0, y(0) = 1;$

(ii) $\begin{cases} \dot{x} = x - 4y + \cos 2t \\ \dot{y} = x + y, \end{cases}$ $x(0) = 1, y(0) = 1;$

(iii) $\begin{cases} \dot{x} = 2x + 2y \\ \dot{y} = 6x + 3y + e^t, \end{cases}$ $x(0) = 0, \dot{x}(0) = 1;$

(iv) $\begin{cases} \dot{x} = 5x - 4y + e^{3t} \\ \dot{y} = x + y, \end{cases}$ $x(0) = 1, y(0) = -1;$

(v) $\begin{cases} \dot{x} = 2x + 5y \\ \dot{y} = -2x + \cos 3t, \end{cases}$ $x(0) = 2, y(0) = -1;$

(vi) $\begin{cases} \dot{x} = x + y + e^{-t} \\ \dot{y} = 4x - 2y + e^{2t}, \end{cases}$ $x(0) = 1, y(0) = -1;$

(vii) $\begin{cases} \dot{x} = 8x + 14y \\ \dot{y} = 7x + y, \end{cases}$ $x(0) = y(0) = 1.$

EXERCISE 8. Write down the general solution and draw the phase portrait for the equation $\dot{\mathbf{x}} = A\mathbf{x}$, when the eigenvalues and eigenvectors of A are as given in each case. You should take particular care with stable (or unstable) nodes to ensure that the trajectories approach (or move away from) the origin tangent to the correct eigenvector.

(i) $\lambda_1 = 1, \mathbf{v}_1 = (1, 1)$ and $\lambda_2 = 2, \mathbf{v}_2 = (1, -1);$

(ii) $\lambda_1 = 1, \mathbf{v}_1 = (1,0)$ and $\lambda_2 = -2, \mathbf{v}_2 = (1,1)$;

(iii) $\lambda_1 = -2, \mathbf{v}_1 = (1,2)$ and $\lambda_2 = -3, \mathbf{v}_2 = (2,-3)$;

(iv) $\lambda_1 = 3, \mathbf{v}_1 = (2,3)$ and $\lambda_2 = -5, \mathbf{v}_2 = (0,1)$;

(v) $\lambda_1 = 3, \mathbf{v}_1 = (1,2)$ and $\lambda_2 = 1, \mathbf{v}_2 = (1,-3)$;

(vi) $\lambda_1 = 2, \mathbf{v}_1 = (0,1)$ and $\lambda_2 = -3, \mathbf{v}_2 = (1,5)$;

(vii) $\lambda_1 = 1, \mathbf{v}_1 = (1,1)$ and $\lambda_2 = 2, \mathbf{v}_2 = (2,1)$;

(viii) $\lambda_1 = -3, \mathbf{v}_1 = (1,3)$ and $\lambda_2 = -1, \mathbf{v}_2 = (-3,2)$.

EXERCISE 9. Find the eigenvalues and eigenvectors of the matrix on the right-hand side in each case, and hence find the coordinate transformation that will decouple the corresponding equations. Show that this transformation has the desired effect. You can also write down the general solution and draw the phase portrait for the equation.

(i) $\dfrac{d\mathbf{x}}{dt} = \begin{pmatrix} 8 & 14 \\ 7 & 1 \end{pmatrix} \mathbf{x}$;

(ii) $\dfrac{d\mathbf{x}}{dt} = \begin{pmatrix} 2 & 0 \\ -5 & -3 \end{pmatrix} \mathbf{x}$;

(iii) $\dfrac{d\mathbf{x}}{dt} = \begin{pmatrix} 11 & -2 \\ 3 & 4 \end{pmatrix} \mathbf{x}$;

(iv) $\dfrac{d\mathbf{x}}{dt} = \begin{pmatrix} 1 & 20 \\ 40 & -19 \end{pmatrix} \mathbf{x}$.

EXERCISE 10. Using the chain rule for $y = y(x(t))$,

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} \Longrightarrow \frac{dy}{dx} = \frac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}}.$$

Therefore, if

$$\frac{dx}{dt} = \lambda_1 x, \quad \frac{dy}{dt} = \lambda_2 y$$

we have

$$\frac{dy}{dx} = \frac{\lambda_2 y}{\lambda_1 x}.$$

Solve the latter equation to determine the curves traced out by trajectories of the system of equations for $x(t)$ and $y(t)$.

EXERCISE 11. Suppose that A has eigenvalues $\lambda_1 = 0, \lambda_2 \neq 0$, with eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$, respectively.

(i) Write down the general solution of $\dot{\mathbf{x}} = A\mathbf{x}$.

(ii) After changing to a coordinate system $\tilde{\mathbf{x}}$ referred to the eigenvectors, we obtain

$$\frac{d\tilde{\mathbf{x}}}{dt} = \begin{pmatrix} 0 & 0 \\ 0 & \lambda_2 \end{pmatrix} \tilde{\mathbf{x}}.$$

By solving these equations draw the phase portrait in the $\tilde{\mathbf{x}}$ system, and sketch the phase portrait for the original coordinates $\mathbf{x}$.

(iii) Draw the phase portrait for the equation $\dfrac{d\mathbf{x}}{dt} = \begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix} \mathbf{x}$.