Chapter 3: part 2

EXERCISE 1. Find the general solution of the following differential equations. You will find it useful to look up Exercise 4 of Chapter 3, part 1. For each item (n), find the unique solution such that $x(0) = n$ and $\dot{x}(0) = 0$. E.g. for item (iii), $x(0) = 3$, etc.

(i) $\ddot{x} - 4x = t^2$;

(ii) $\ddot{x} - 4\dot{x} = t^2$;

(iii) $\ddot{x} + \dot{x} - 2x = 3e^{-t}$;

(iv) $\ddot{x} + \dot{x} - 2x = e^{t}$;

(v) $\ddot{x} + 2\dot{x} + x = e^{-t}$;

(vi) $\ddot{x} + \omega^2 x = \sin(\alpha t)$ for $\alpha \neq \omega$;

(vii) $\ddot{x} + \omega^2 x = \sin(\alpha t)$ for $\alpha = \omega$;

(viii) $\ddot{x} + 2\dot{x} + 10x = e^{-t}$;

(ix) $\ddot{x} + 2\dot{x} + 10x = e^{-t}\cos(3t)$;

(x) $\ddot{x} + 6\dot{x} + 10x = e^{-3t}\cos t$,

(xi) $\ddot{x} + 4\dot{x} + 4x = e^{2t}$;

EXERCISE 2. Find a particular solution for $\ddot{x} + \dot{x} - 2x = 12e^{-t} - 6e^{t}$. You may find useful parts (iii) and (iv) of the previous exercise.

EXERCISE 3. Suppose that $x(t)$ denotes the distance of some system from its equilibrium position, such that it satisfies the differential equation

$$\ddot{x} + \omega^2 x = a\cos\alpha t,$$

where the second term on the left hand side accounts for harmonic oscillations and the right hand side is an external forcing that is periodic in time. Discuss the form of the general solution in terms of the natural frequency of oscillation ω and the frequency α of the external force.

EXERCISE 4. Consider the oscillating system of the previous exercise, but now in the presence of a non-negligible viscous drag so that Newton's second law reads

$$\ddot{x} + \lambda\dot{x} + \omega^2 x = a\cos\alpha t,$$

in the underdamped case in which $\lambda^2 < 4\omega^2$. Show that $x_p(t) = M\cos(\alpha t - \phi)$ is a particular solution, finding its amplitude M and phase ϕ in terms of the system parameters. Discuss how the relative ratio between the system response and the external forcing, M/a, behaves with the external frequency α and compare with the zero-drag ($\lambda = 0$) case.

EXERCISE 5. Working by analogy with the second order case, find the general solution of the following inhomogeneous higher-order linear equations with constant coefficients:

(i) $\dfrac{d^3x}{dt^3} - 6\dfrac{d^2x}{dt^2} + 11\dfrac{dx}{dt} - 6x = e^{-t}$,

(ii) $y''' - 3y' + 2 = \sin x$,

(iii) $\dfrac{d^4x}{dt^4} - 4\dfrac{d^3x}{dt^3} + 8\dfrac{d^2x}{dt^2} - 8\dfrac{dx}{dt} + 4x = \sin t$.

EXERCISE 6. Use the method of variation of constants to find a particular solution for the following equations:

(i) $y'' - y' - 6y = e^x$;

(ii) $\ddot{x} - x = t^{-1}$ (leave the answer as an integral);

(iii) $y'' + 4y = \cot 2x$. Use that $\int \csc x\, dx = \ln|\csc x - \cot x|$;

(iv) $t^2\ddot{x} - 2x = t^3$ (to find the solutions of the homogeneous equation, try solutions of the form $x = t^k$ with k a parameter);

(v) $\ddot{x} - 4\dot{x} = \tan t$ (leave your answer as an integral).

EXERCISE 7. Show that the solution of the initial value problem

$$L[y] = y'' + p(t)y' + q(t)y = g(t), \quad y(t_0) = y_0, \quad y'(t_0) = y_0'$$

can be written as $y(t) = u(t)+v(t)$, where u and v are solutions of two different initial value problems:

$$\begin{array}{lll} L[u] = 0, & u(t_0) = y_0, & u'(t_0) = y_0', \\ L[v] = g(t), & v(t_0) = 0, & v'(t_0) = 0. \end{array}$$

EXERCISE 8. The so-called method of reduction of order can be used to find one solution of the inhomogeneous equation $L[y] = y'' + p(t)y' + q(t)y = g(t)$, provided one solution $y_1(t)$ of the corresponding homogeneous equation is known. Indeed, writing $y = v(t)y_1(t)$, show that $y(t)$ satisfies $L[y] = g$ if v is a solution of

$$y_1(t)v'' + [2y_1' + p(t)y_1]v' = g(t).$$

This is a first order linear equation for $v'(t)$. Solving this equation, integrating the result, and then multiplying by $y_1(t)$ leads to the general solution of the problem for $y(t)$.

EXERCISE 9. For $t > 0$ the function $y_1(t) = 1/t$ is a solution of $t^2\ddot{y} - 2t\dot{y} - 4y = 0$. Using the previous exercise, find a second, linearly independent solution.
