Wronksian and linear independence

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Let $I \subset \mathbb{R}$ be an open interval, and let x_1 and x_2 be solutions of the following second-order, linear, homogeneous ODE

$$x'' + p(t)x' + q(t)x = 0 \quad \text{for all } t \in I. \tag{1}$$

Define the *Wronskian* of x_1, x_2 in I as

$$W(x_1, x_2) = \det \begin{pmatrix} x_1 & x_2 \\ x_1' & x_2' \end{pmatrix} = x_1' x_2 - x_1 x_2' \,.$$

Lemma 0.1. *Two nontrivial solutions x_1, x_2 of (1) are linearly dependent if and only if $W(x_1, x_2) \equiv 0$ in I.*

Proof. Let us suppose that x_1, x_2 are linearly dependent, that is, that there exists a $c \neq 0$ such that $x_1 = cx_2$. Therefore

$$W = x_1' x_2 - x_1 x_2' = c x_2' x_2 - c x_2 x_2' = 0 \,.$$

Now let us assume that $W \equiv 0$ in I. Let us fix $t_0 \in I$, and we have

$$\det \begin{pmatrix} x_1(t_0) & x_2(t_0) \\ x_1'(t_0) & x_2'(t_0) \end{pmatrix} = 0 \,,$$

so there exists $\vec{c} = (c_1, c_2)^T \neq \vec{0}$ such that

$$\begin{cases} c_1 x_1(t_0) + c_2 x_2(t_0) = 0 \\ c_1 x_1'(t_0) + c_2 x_2'(t_0) = 0 \,. \end{cases}$$

Now, for $t \in I$, let us define $x = c_1 x_1 + c_2 x_2$. By the superposition principle, and by construction, we have

$$\begin{cases} x'' + p(t)x' + q(t)x = 0 \,, & t \in I \\ x(t_0) = 0 \,, \\ x'(t_0) = 0 \,, \end{cases}$$

so by the uniqueness theorem, $x = c_1 x_1 + c_2 x_2 = 0$ for all $t \in I$. Therefore x_1 and x_2 are linearly dependent. □