Diff Eq

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Differential equations: eq involving an unknown function (ind var) and its derivatives

ODE: diff. eq. in which the unknown function depends on only one ind var

Linear ODE: $a_0(x)y^{n}(x) + \dots + a_n(x)y(x) = b(x)$

Ex (ODE) ▸ $x'(t) = x(t)$ (dep var / ind var) ▸ $x(t) = e^t$

PDE: diff. eq. in which the unknown function depends on more than one variables

Ex (PDE) ▸ $\frac{du}{dt} = \frac{d^2u}{dx^2}$ ← "Heat eq."

Order: order of the highest derivative in the ODE

Linear / Non-linear ODE's

Consider u_1 & u_2, two solutions to an ODE.
Then the ODE is linear for all $\alpha, \beta \in \mathbb{R}$

$\alpha u_1 + \beta u_2$ is also a solution to the ODE

If not, the the ODE is non-linear

Check linearity of $u' = u$?

Call $u = \alpha u_1 + \beta u_2$ (solutions ⇒ $u_1' = u_1$ & $u_2' = u_2$)

then $u' = \alpha u_1' + \beta u_2' = \alpha u_1 + \alpha u_2 = u$

Linear

Check linearity $u'' = (u')^2 + 1$

let $v = \alpha u_1 + \beta u_2$

$v' = \alpha u_1' + \beta u_2'$

$v'' = \alpha u_1'' + \beta u_2''$

▸ $\alpha u_1'' + \beta u_2' = (\alpha u_1' + \beta u_2')^2 + 1$

▸ $\alpha(u_1'^2 + 1) + \beta(u_2'^2 + 1) = \alpha^2 u_1'^2 + 2\alpha\beta u_1' u_2' + \beta^2 u_2'^2 + 1$

Non linear

Systems of ODE's: two or more dependent variables

ex $\begin{cases} x'(t) = 2x(t) - y(t) \\ y'(t) = -x(t) + 3y(t) \end{cases}$

EXAMPLES

1. Population Growth / Decay

$x(t)$ is the population @ t
≥ 0 for all t
The rate of change of x is proportional to x itself with prop c $k \in \mathbb{R}$

⇒ ODE: $x'(t) = kx(t)$ (1st order, linear)

Let initial population size = $x(0) > 0$

▸ $\frac{dx}{dt} = kx \blacktriangleright \int_0^t \frac{dx}{x} = \int_0^t k\,dt \blacktriangleright \ln\frac{x}{x_0} = kt \blacktriangleright x = x_0 e^{kt}$

or

$\frac{x'(t)}{x(t)} = k \blacktriangleright \int_0^t \frac{x'(\tau)}{x(\tau)}\,d\tau = \int_0^t k\,d\tau \blacktriangleright \int_0^t (\ln(x(\tau)))'\,d\tau = kt$ ($x(\tau) > 0$)

▸ $\ln\left(\frac{x}{x_0}\right) = kt \blacktriangleright \frac{x}{x_0} = e^{kt} \blacktriangleright x = x_0 e^{kt}$

if $k > 0$ pop ↑
if $k < 0$ pop ↓

In reality, population doesn't grow exponentially forever, but reaches a flat max, we use a logistic correction

2. Newtons 2nd Law "F = ma"

Free fall of an object under g

ODE ▸ $-mg = my(t)'' \blacktriangleright$ 2nd order linear

$\frac{d^2y}{dt^2} = -g \blacktriangleright y = -\iint g\,dt\,dt = -\frac{1}{2}gt^2$

INITIAL VALUE PROBLEMS

We are looking for an existence of solution as well as a uniqueness of solution

An IVP for a first order ODE is:

$$\begin{cases} y'(x) = f(x, y(x)) \\ y(a_0) = x_0 \end{cases}$$

PICARD TH. $\left[y'(x) = f(x, y(x)) \right]$

Let f, $\frac{df}{dy}$ be cont. func of x and y near the initial value $(x_0, y(x_0))$

Then there exists $h > 0$ such that the IVP $\begin{cases} y'(x) = f(x, y(x)) \\ y(x_0) = y_0 \end{cases}$

has a unique solution for $|x - x_0| < h$.

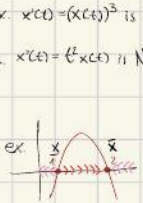
Stationary Points: points in "x" where $f(x) = 0$

- if $f(x) = 0$ & $f'(x) > 0$, then x is unstable
- if $f(x) = 0$ & $f'(x) < 0$, then x is a Stable Stationary Pt.

Autonomous 1st order ODE's: When "f" only depends on depvar "x"

ex. $x'(t) = (x(t))^3$ is Autonomous bc $f(t) = x^3$ doesn't dep directly on "t"

ex. $x'(t) = t^2 x(t)$ ⇒ Not Autonomous because $f(x,t) = t^2 x$

ex. 

pt 1 is unstable, since $f' > 0$

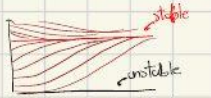
pt 2 is stable bc $f' < 0$

$x'(t) = f(x(t))$

- if $x_0 = \underline{x} \Rightarrow x'(0) = f(\underline{x}) = 0$
- if $x_0 = \bar{x} \Rightarrow x'(0) = f(\bar{x}) = 0$
- if $\underline{x} < x_0 < \bar{x}$, $f(x_0) > 0 \Rightarrow x' > 0$ (going right)
- if $x_0 < \underline{x}$ or $x_0 > \bar{x}$, $f(x_0) < 0 \Rightarrow x' < 0$ (going left)

• concrete example.

$$\begin{cases} x'(t) = Kx\left(1 - \frac{x}{M}\right) = kx - \frac{k}{M}x^2 \leftarrow \text{nonlinear} \\ x(0) = x_0 \geq 0 \end{cases} \quad k, M > 0$$



* each curve is a solution to the ode. Poin 0 is unstable because slight variation generates M.

If $x_0 = 0$, x stays 0

If $x_0 \neq 0$, x ends up at M

Critical points → $x = 0$ & $x = M$

① $x = 0$ ▷ $f' = K + 0 > 0$ → unstable

② $x = M$ ▷ $K - 2K < 0$ → stable

EXPLICIT SOLUTIONS TO SOME 1st Order ODE's

1. Separable Equations (4-6) Sheet 2

If $f \neq 0$ (never), then divide both sides by f, separating x & t

$$\begin{cases} x'(t) = f(x(t))g(t) \\ x(t_0) = x_0 \end{cases} \blacktriangleright \int_{t_0}^{t} \frac{x'(\tau)}{f(x(\tau))} d\tau = \int_{t_0}^{t} g(\tau) d\tau$$

Ex. $x'(t) = Kx(t) \Rightarrow x(t) = Ce^{Kt}$

Ex. $\begin{cases} x'(t) = Kx(t)\left(1 - \frac{x(t)}{M}\right) \\ x(t_0) = t_0 \end{cases} \blacktriangleright \int_{t_0}^{t} \frac{dx}{x(1 - \frac{x}{M})} = \int_{t_0}^{t} K dt = K(t - t_0)$

$$\int_{x_0}^{x} \frac{1}{x(1 - \frac{x}{M})} dx = \int_{x_0}^{x} \left(\frac{1}{x} + \frac{1}{M - x} \right) dx = \ln\left(\frac{x}{x_0}\right) + \ln\left(\frac{M - x_0}{M - x}\right) = \ln\left(\frac{x(M - x_0)}{x_0(M - x)}\right)$$

$$\frac{1}{x(1 - \frac{x}{M})} = \frac{A}{x} + \frac{B}{1 - \frac{x}{M}} \quad x = 0 \Rightarrow 1 = A \quad x = M \Rightarrow 1 = \frac{B}{M} \Rightarrow B = M$$

$$\ln\left(\frac{x(M - x_0)}{x_0(M - x)}\right) = K(t - t_0) \Rightarrow \frac{x(M - x_0)}{x_0(M - x)} = e^{K(t - t_0)}$$

$$x = M - x_0 + x(1 + \frac{x_0}{M - x_0}) = M - x_0 + x\left(\frac{M}{M - x_0}\right)$$

$$x = \frac{\frac{Mx_0 e^{K(t-t_0)}}{M - x_0}}{1 + \frac{x_0 e^{K(t-t_0)}}{M - x_0}}$$

$$x = \frac{Mx_0 e^{K(t-t_0)}}{M - x_0 + x_0 e^{K(t-t_0)}} = \frac{Mx_0}{x_0 + (M - x_0)e^{-K(t-t_0)}} \quad \lim_{t \to \infty} = \frac{Mx_0}{x_0 + 0} = M$$

2. Using integrating factors

$x'(t) + q(t)x(t) = g(t)$ ← it is not separable

$I(t)x'(t) + I(t)q(t)x(t) = I(t)g(t)$ ← multiply equation by unknown factor $I(t)$

▶ $(I(t)x(t))' = I(t)x'(t) + I(t)x(t) = I(t)x'(t) + I(t)q(t)x(t)$ ← we want to choose $I(t)$ so that the left hand side of the equation is equal to $(I(t)x(t))'$

we are left with $I'(t) = I(t)q(t) \Rightarrow \frac{I'(t)}{I(t)} = q(t)$ ▶ $\ln(I(t)) = \int q(t)dt + \tilde{P}(t)$ ← primitive of $q(t) \Rightarrow e.g. \int_{t_0}^{t} q(\tau)d\tau = \tilde{P}(t) - \tilde{P}(t_0)$

▶ $I(t) = e^{\tilde{P}(t)}$ ← **INTEGRATING FACTOR**

▶ $(I(t)x(t))' = I(t)g(t)$ ▶ integrate both sides ▶ $I(t)x(t) = \int I(t)g(t)dt$ ← now we divide by $I(t)$

▷ $x(t) = \frac{1}{I(t)} \int I(t)g(t)dt$ → general solution to the ODE

Now, lets consider IVP's $\begin{cases} x'(t) + q(t)x(t) = g(t), & t > t_0 \\ x(t_0) = x_0 \end{cases}$

STEP 1▷ find $I(t)$, we define $\tilde{P}(t) = \int_{t_0}^{t} q(\tau)d\tau$ ▶ notice $\tilde{P}(t_0) = \int_{t_0}^{t_0} q(t)dt = 0$ ▶ therefore $I(t) = e^{\tilde{P}(t)}$, $I(t_0) = e^0 = 1$

STEP 2▷ With the knowledge of $I(t)$, we integrate from $t_0 \to t$ ▶ $\int_{t_0}^{t} (I(\tau)x(\tau))'d\tau = \int_{t_0}^{t} I(\tau)g(\tau)d\tau$

S3. simplify ▶ $I(t)x(t) - \underbrace{I(t_0)}_{1}\underbrace{x(t_0)}_{x_0} = I(t)x(t) - x_0 = \int_{t_0}^{t} I(\tau)g(\tau)$

SOLUTION ▶ $x(t) = \frac{x_0}{I(t)} + \frac{1}{I(t)} \int_{t_0}^{t} I(\tau)g(\tau)d\tau$, where $I(t) = e^{\tilde{P}(t)}$

EXACT EQUATIONS

- Let y be a function of x
- Lets consider the function $F(x,y) = C$, where C is a constant
- Lets take the derivative: $\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\cdot\frac{dy}{dx} = 0$, this is a 1st Order ODE

▸ Therefore, if we have an ODE of the form $f(x,y) + g(x,y)\frac{dy}{dx} = 0$ & we find $F(x,y)$ such that $\frac{\partial F}{\partial x} = f$ $\frac{\partial F}{\partial y} = g$

then, the solution to the ODE is $F(x,y) = C$

an observation we find is $\frac{\partial f}{\partial y} = \frac{\partial^2 F}{\partial x \partial y} = \frac{\partial g}{\partial x}$, so we can replace the previus condition by this one

THEOREM: Let $f, g, \frac{\partial f}{\partial y}, \frac{\partial g}{\partial x}$ be cont funct then $f(x,y) + g(x,y)\frac{dy}{dx} = 0$ is an exact ODE of $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$
where the solution is $F(x,y) = C$ and $\frac{\partial F}{\partial x} = f$, $\frac{\partial F}{\partial y} = g$

ex.1. $\overbrace{x^3 + \frac{y}{x}}^{f} + \overbrace{(y^2 + \ln x)}^{g}\frac{dy}{dx} = 0$

↑ the ode is not linear due to the $y^2\frac{dy}{dx}$

$\frac{\partial f}{\partial y} = \frac{1}{x}$, $\frac{\partial g}{\partial x} = \frac{1}{x}$

→ $\int f\,dx = \int g\,dy$

▸ $\int (x^3 + \frac{y}{x})\,dx = \frac{x^4}{4} + y\ln x + p'(y)$

▸ $g = \frac{\int f\,dx}{dy} = \ln x' + p'(y) = y^2 + \ln x + p(y) = \frac{y^3}{3} + C$

▸ $F(x,y) = \frac{x^4}{4} + \frac{y^3}{3} + y\ln x = C$

implicit sol, since x & y are together

ex.2. $\overbrace{(1 + e^x y + xe^x y)}^{f} + \overbrace{(xe^x + 2)}^{g}\frac{dy}{dx} = 0$

$\frac{\partial f}{\partial y} = e^x + xe^x$

$\frac{\partial g}{\partial x} = e^x + xe^x$ ($\frac{g-\int g}{x}$, $\frac{g'}{x'}$)

→ $\int f\,dx = x + e^x y + y\int xe^x = x + e^x y + yxe^x - ye^x + p(y)$

→ $x + xe^x - e^x + p(y) = xe^x + 2$ ▸ $p'(y) = 2$ ▸ $p(y) = 2y + C$

▸ $F(x,y) = x + yxe^x + 2y = C$

▸ $y = \frac{C - x}{xe^x + 2}$

2nd Order Linear ODE's

$$\rightarrow x''(t) + p(t)x'(t) + q(t)x(t) = f(t)$$

- For example, the equation of a spring: $p(t) \rightarrow$ friction of the system; $q(t) \rightarrow$ spring constant; $f(t) \rightarrow$ forcing

- We have a basic well-posedness (existance and uniqueness) result in the style of "Picard"

THEOREM

Given $\ddot{x} = f(\dot{x}, x, t)$, if f, $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial \dot{x}}$ are continuous in a cube $\{(a_1, a_2) \times (b_1, b_2) \times (t_1, t_2)\}$, then the IVP with $x(t_0) = x_0$ and $\dot{x}(t_0) = y_0$ has a unique solution $\begin{pmatrix} t \in (t_1, t_2) \\ x \in (a_1, a_2) \\ y \in (b_1, b_2) \end{pmatrix}$ (if ...)

SUPERPOSITION PRINCIPLE: if x_1 & x_2 are solutions to the linear homogeneous ODE $(x'' + px' + qx = 0)$, then, any linear combination of x_1 & x_2 $\triangleright$ $\alpha x_1 + \beta x_2$ is also a solution

REMARK: If $f(t) = 0 \rightarrow$ Homogeneous, if $f(t) \neq 0 \rightarrow$ unhomogeneous.

DEF: The functions $x_1(t), x_2(t), \ldots, x_n(t)$ are linearly independent on an interval I if the only solution for $\alpha x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n = 0$ if only if $\alpha_1 = \alpha_2 = \alpha_3 = \ldots = \alpha_n$

THEOREM

Any solution to $x'' + px' + qx = 0$ can be written as a linear combination $\{x(t) = C_1 x_1(t) + C_2 x_2(t)\}$, where x_1 and x_2 are linearly indep. sol's. to the same problem

- By the existance and uniqueness th it is easy to construct two lin. indep solutions.

EXAMPLE

$$(IPV)\begin{cases} x'' + px' + qx = 0 \\ x = x_0 \\ x' = y_0 \end{cases} @ t=0 \quad \triangleright \begin{pmatrix} x_1(t_0) & x_2(t_0) \\ x_1'(t_0) & x_2'(t_0) \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

↳ Only a solution when the determinant $\neq 0$

* x_0 & y_0 can't vary.

* We want to prove $x(t) = C_1 x_1(t) + C_2 x_2(t)$

Before Continuing, CONSIDER → "WRONSKIAN"

$$W(x_1, x_2)(t) = x_1(t)x_2'(t) - x_2(t)x_1'(t)$$

→ We find a solution for C_1 & C_2 if $W(x_1, x_2) \neq 0$

(Which equation does this coincide ?)

$\triangleright W = x_1x_2' + x_1x_2'' - x_2x_1' - x_2x_1'' = x_1(-px_2' - qx_2) - x_2(-px_1' - qx_1) = -p(x_1x_2' - x_2x_1') = -pW$

$\triangleright W(x) = -p(x)W(x)$

$W(x) = Ce^{-\int p(x)dx}$, and by Picard's Th either $\begin{cases} W = 0 \text{ (obviously)} \\ W \neq 0 \text{ (never)} \end{cases}$

either 0 for all t, or $\neq 0$ for all t

Homogeneous, linear 2nd degree ODE's with constant $p(t)$ & $q(t)$

→ Consider $x'' + px' + qx = 0$ for $p, q \in \mathbb{R}$

→ We'll use the prev Th to find general solutions to this problem, since we know that any solution can be written as a lin. combination.

→ We look to solutions of the type $x(t) = e^{kt}$, for k to be determined

We have $(e^{kt})' + p(e^{kt})' + qe^{kt} = 0 \triangleright k^2e^{kt} + pke^{kt} + qe^{kt} = (k^2 + pk + q)e^{kt} = 0$

Since $e^{kt} \neq 0$, then: $k^2 + pk + q = 0$, $k = \frac{-p \pm \sqrt{p^2 - 4q}}{2}$

CASE 01 (2 real roots): $\begin{cases} x_1(t) = e^{k_1 t} \\ x_2(t) = e^{k_2 t} \end{cases} \rightarrow x(t) = C_1e^{k_1 t} + C_2e^{k_2 t}$

CASE 02 (1 real root): $\triangleright x(t) = C_1e^{kt} + C_2te^{kt}$

CASE 03 (2 complex sols): We have $k_1, k_2 = \frac{-p}{2} \pm \left(\frac{\sqrt{4q - p^2}}{2}\right)i$ $\triangleright$ $x(t) = C_1e^{(\alpha + i\omega)t} + C_2e^{(\alpha - i\omega)t}$

$= C_1e^{\alpha t}(\cos(\omega t) + i\sin(\omega t)) + C_2e^{\alpha t}(\cos(\omega t) + i\sin(\omega t))$

$\triangleright (C_1 + C_2)e^{\alpha t}\cos\omega t + i(C_1 - C_2)e^{\alpha t}\sin\omega t$

↳ If we choose $C_1 = \frac{C_1 - C_2}{i}$ (complex conj of C_1) $\triangleright x(t) = e^{\alpha t}(A\cos\omega t + B\sin\omega t)$

2nd order linear unhomogeneous ODE's w/ cte coeffs

✱ $X''(t) + pX'(t) + qX(t) = f(t)$

The general solution for ✱ $(X(t))$ is the sum of the associated homo problem $(f(t) \equiv 0)(X_H(t))$ and a particular solution for ✱ $(X_N(t))$

$$\triangleright\ X(t) = X_H(t) + X_N(t)$$

In order to find $X_N(t)$, well have to do an educated guess + trial and error

We base our guess on $f(t)$

For example: if $f(t)$ is an N^{th} order polynomial $\triangleright X_N(t) = C_n t^n + C_{n-1}t^{n-1} + \dots + C_2 t^2 + Ct + C$

Then we plug $X_N(t)$ on the full equation and solve for C_i

What if $f(t) = Ce^{\alpha t}$

→ Case 1: If α is not a root of $k^2 + pk + q = 0$, then $X_N(t) = Ae^{\alpha t}$

→ Case 2: If α is a root of $k^2 + pk + q = 0$, then $X_N(t) = Ate^{\alpha t}$

What if $f(t) = \alpha\cos(\omega t) + \beta\sin(\omega t)$ *

then, $X_N(t) = A\cos(\omega t) + B\sin(\omega t)$

* if it is already in $X_H(t)$, then $X_N(t) = At\cos(\omega t) + Bt\sin(\omega t)$

Finally, if $f(t)$ is composed by the sum of two or more terms, we may be able to solve for X_N by "decomposition"

This procedure also works for higher order diff.eq

→ get n roots and get $X_H(t)$ of the form of $C_n e^{kt}$

→ Find $X_N(t)$ in the previous methods.

Proof: Take $\Phi(t) = X(t) - X_N(t)$ • $\begin{cases} \Phi' = x' - x_N' \\ \Phi'' = x'' - x_N'' \end{cases}$

$\Rightarrow \Phi'' + p\Phi' + q\Phi = x'' - x_N'' + p(x' - x_N') + q(x - x_N)$

$= (x'' + px' + qx) - (x_N'' + px_N' + qx_N)$

$= f(t) - f(t) = 0$

Since Φ satisfies the homo eq $\triangleright \Phi(t) = X_H(t) = X(t) - X_N(t) \triangleright X(t) = X_H(t) - X_N(t)$

Find general solution for: $X'' + X' - 6x = 12$

Step 1: Find $X_H(t)$ $\triangleright p=1, q=-6, f(t)=12$ $\triangleright$ Char. poly $= k^2 + pk + q = \boxed{k^2 + k - 6 = 0}$

$\rightarrow (k+3)(k-2) \triangleright k = -3, 2 \Rightarrow X_H(t) = C_1e^{-3t} + C_2e^{2t}$

Step 2: Find $X_N(t)$ $\triangleright$ Define $X_N(t) = C$ $\triangleright$ Plug in $\triangleright 0 + 0 - 6C = 12$

then $C = \boxed{X_N(t) = -2}$ $\triangleright$ General Solution $\Rightarrow X(t) = C_1e^{-3t} + C_2e^{2t} - 2$

What about if $f(t) = 3t$? $\triangleright X_N(t) = Ct + C_0 \triangleright 0 + C_1 - 6Ct - 6C_0 = 3t$

$\left\{\begin{array}{l} -6Ct = 3t \triangleright \boxed{C_1 = -6} \\ C_1 - 6C_0 = 0 \triangleright \boxed{C_0 = \frac{C_1}{6} = -1} \end{array}\right\} X_N(t) = -6t - 1 \triangleright X(t) = C_1e^{-3t} + C_2e^{2t} - 6t - 1$

What about $f(t) = 4e^{-2t}$? $\triangleright X_N = Ae^{-2t} \triangleright 4Ae^{-2t} - 2Ae^{-2t} - 6Ae^{-2t} = 4e^{-2t}$

$\triangleright -4Ae^{-2t} = 4e^{-2t} \triangleright A = -1 \triangleright X_N(t) = -e^{-2t}$

2nd Order Linear ODE's w/ variable coeff

$$x''(t) + p(t)x'(t) + q(t)x(t) = f(t)$$

In general, we won't be able to find all solutions, but we will study two different strategies

1. REDUCTION OF ORDER

This method is for finding a second solution to a homogeneous problem if one is already known

- Suppose we have a solution $u(t)$ for $x'' + p(t)x' + q(t)x = 0$, we look for a second solution of the form $x(t) = y(t)u(t)$

> Plug $x(t)$ into the equation $\rightarrow [y''u + 2y'u' + yu''] + p(t)[y'u + yu'] + q(t)[yu] = 0$

$\rightarrow [u'' + p(t)u' + q(t)u]y + [2u' + p(t)u]y' + y''u = 0$ (the $[u'' + p(t)u' + q(t)u]$ term $= 0$)

> We define $z = y'$ and notice z satisfies a 1st Order Equation

VARIATION OF CONSTANTS

· To find a particular solution to the complete equation $x'' + p(t)x' + q(t)x = f(t)$ if two linearly independent solutions to the homogeneous equation (x_1, x_2) are known

Idea: We look for solution $X(t) = u_1(t)x_1(t) + u_2(t)x_2(t)$, where we need to solve for u_1 & u_2

$X' = u_1'x_1 + u_1x_1' + u_2'x_2 + u_2x_2'$

We impose $\rightarrow u_1'x_1 + u_2'x_2 = 0 \rightarrow \boxed{u_1' = -u_2'\frac{x_2}{x_1}}$ *

$\Rightarrow X' = u_1x_1' + u_2x_2'$

$\Rightarrow X'' = u_1'x_1' + u_1x_1'' + u_2'x_2' + u_2x_2''$

$(u_1'x_1' + u_1x_1'' + u_2'x_2' + u_2x_2'') + p(t)(u_1x_1' + u_2x_2') + q(t)(u_1x_1 + u_2x_2) = f(t)$

$u_1(x_1'' + p(t)x_1' + q(t)x_1) + u_2(x_2'' + p(t)x_2' + q(t)) + u_1'x_1' + u_2'x_2' = f(t)$

$\rightarrow u_1'x_1' + u_2'x_2' = f(t)$, since x_1 & x_2 satisfy the homogeneous eq.

* $(-u_2'\frac{x_2}{x_1})x_1' + u_2'x_2' = u_2'(x_2' - \frac{x_2x_1'}{x_1}) = f(t) \rightarrow u_2' = \frac{f}{x_2' - \frac{x_2x_1'}{x_1}} = u_2'\left(\frac{x_1x_2' - x_1'x_2}{x_1}\right)$

$\rightarrow u_2' = \frac{f \cdot x_1}{W}$ or $u_1' = \frac{f \cdot x_2}{W}$ where $W = x_1x_2' - x_2x_1'$

$u_1 = \int \frac{x_2 f}{W}$

$u_2 = \int \frac{x_1 f}{W}$

SYSTEMS OF DIFF. EQ

- We have n dependent variables and 1 indep var (t)
- Dependant Variables are $x_1, x_2, x_3, \dots, x_n$
- They satisfy a system of ODE's

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2, \dots, x_n, t) \\ \dot{x}_2 = f_2(x_1, x_2, \dots, x_n, t) \\ \vdots \\ \dot{x}_n = f_3(x_1, x_2, \dots, x_n, t) \end{cases}$$

Notation $\begin{cases} \vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = (x_1, \dots, x_n)^T \\ \vec{f} = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} = (f_1, \dots, f_n)^T \\ \vec{x} \text{ is a function of } t \\ \vec{f} \text{ is a function of } \vec{x} \text{ and } t \end{cases} \Rightarrow \dot{\vec{X}} = \vec{f}(\vec{x}, t)$

If $\vec{f}$ and $D_{\vec{x}}\vec{f}$ (Jacobian Matrix) are cont functions in $\vec{x}$ and t in an open neighborhood of $\vec{x}_0$ and t_0, then the IVP

$$\begin{cases} \dot{\vec{x}} = \vec{f}(\vec{x}, t) \\ \vec{x}(t_0) = \vec{x}_0 \end{cases}$$ has a unique local solution

Important Point: ODE's of order $n \geq 2$ can be converted into a system of n first order ODE's

Ex. $\ddot{x} = f(\dot{x}, x, t) \Rightarrow \begin{cases} \dot{x}_1 = \dot{x} = x_2 \\ \dot{x}_2 = \ddot{x} = f(x_2, x_1, t) \end{cases}$

Special Important Case: Linear Systems

$$\begin{cases} \dot{\vec{x}} = A(t)\vec{x} + \vec{f}(t) \\ \vec{x}(t_0) = \vec{x}_0 \end{cases}$$

A is an $n \times n$ matrix (dep on t)

$\vec{f}$ is a vectorial fnction of t ▸ If $\vec{f} = 0$ (homogeneous)

Existence and Uniqueness are guaranteed if $A, \vec{f}$ are cont in $\vec{x}_0$ & t_0

Notation If we have two solutions $\vec{x}_1, \vec{x}_2$ ▸ We use $\vec{x}_1 = \begin{pmatrix} x_{11} \\ x_{21} \\ x_{n1} \end{pmatrix}$, $\vec{x}_2 = \begin{pmatrix} x_{12} \\ x_{22} \\ x_{n2} \end{pmatrix}$

- Consider the linear homogeneous system $\dot{\vec{X}} = A(t)\vec{X}$, if $\vec{x}_1, \vec{x}_2, \dots \vec{x}_k$ are solutions, so is the lin. combination $C_1\vec{x}_1 + C_2\vec{x}_2 + \dots + C_k\vec{x}_k$

Theorem: The solutions $\vec{x}_1 \dots \vec{x}_k$ to $\dot{\vec{x}} = A(t)\vec{x}$ are lin.indep if and only if the Wronskian ▸ $W(\vec{x}_1, \dots, \vec{x}_k) \neq 0 \quad \forall t \in I$

$$W(\vec{x}_1, \dots, \vec{x}_k) = \det \begin{pmatrix} x_{11} & \cdots & x_{1k} \\ \vdots & & \vdots \\ x_{k1} & \cdots & x_{kk} \end{pmatrix}$$

Focus on $n=2$ (system of 2ODE's) ▸ our system $\dot{\vec{x}} = A(t)\vec{x} + \vec{f}(t)$ can be written as $\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix}$

A fundamental set of solutions to this problem are: $\begin{cases} \vec{x}_1(t) = \begin{pmatrix} x_{11}(t) \\ x_{21}(t) \end{pmatrix} \\ \vec{x}_2(t) = \begin{pmatrix} x_{12}(t) \\ x_{22}(t) \end{pmatrix} \end{cases}$ such that $W(\vec{x}_1, \vec{x}_2) \neq 0$

Theorem: Consider the linear system $\dot{\vec{x}} = A(t)\vec{x} + \vec{f}(t)$, the general solution $\vec{x}$ can be written as the sum $\vec{x} = \vec{x}_h + \vec{x}_p$, where $\begin{cases} \vec{x}_h = \text{the solution to } \dot{\vec{x}} = A(t)\vec{x} \\ \vec{x}_p = \text{the particular sol to } \dot{\vec{x}} = A(t)\vec{x} + \vec{f}(t) \end{cases}$

Example

$$\begin{cases} \dot{x} = ax + by \\ \dot{y} = cx + dy \end{cases} \Rightarrow y = \frac{\dot{x} - ax}{b} \Rightarrow \dot{y} = \frac{\ddot{x} - a\dot{x}}{b}$$

$$\Rightarrow \frac{\ddot{x} - a\dot{x}}{b} = cx + d\frac{\dot{x} - ax}{b} \Rightarrow \ddot{x} - a\dot{x} = bcx + d\dot{x} - adx \Rightarrow \ddot{x} - (a+b)\dot{x} + (ad - bc)x = 0$$

second order diff eq

Method 2 - Lin Algebra

Look for solution of $\dot{x} = Ax$ in the form $x(t) = e^{\lambda t}\vec{v}$

$\vec{v} \neq 0$: some vector to be determined

$\lambda \neq 0$: some number to be determined

$$\Rightarrow \lambda e^{\lambda t}\vec{v} = Ae^{\lambda t}\vec{v} \Rightarrow \lambda\vec{v} = A\vec{v}$$ ← same as an eigenvalue problem, $\vec{v}$ associated eigenvector

we need to solve $\det(A - \lambda I) = 0$

Example. $A = \begin{pmatrix} 2 & 2 \\ 6 & 3 \end{pmatrix} \Rightarrow \det(A - \lambda I) = \det\begin{pmatrix} 2-\lambda & 2 \\ 6 & 3-\lambda \end{pmatrix} = (2-\lambda)(3-\lambda) - 12 = \lambda^2 - 5\lambda - 6 = 0 \Rightarrow \begin{cases} \lambda = 6 \\ \lambda = -1 \end{cases}$ ($(\lambda-6)(\lambda+1)$)

find eigenvectors

$\lambda = 6 \Rightarrow (A - 6I)\vec{v}_{\lambda=6} = 0 \Rightarrow \left(\begin{array}{cc|c} -4 & 2 & 0 \\ 6 & -3 & 0 \end{array}\right) = \left(\begin{array}{cc|c} -4 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow -4\alpha + 2\beta = 0 \Rightarrow \beta = 2\alpha \Rightarrow \vec{v}_{\lambda=6} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\lambda = -1 \Rightarrow (A + I)\vec{v}_{\lambda=-1} = 0 \Rightarrow \left(\begin{array}{cc|c} 3 & 2 & 0 \\ 6 & 4 & 0 \end{array}\right) = \left(\begin{array}{cc|c} 3 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow 3\alpha + 2\beta = 0 \Rightarrow \vec{v}_{\lambda=-1} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$

* for diagonal $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ and upper triangular $\begin{pmatrix} a & c \\ 0 & b \end{pmatrix}$ matrices, the eigenvalues are a, b

Three Cases for eigenvalues

▸ **Case 1: Two Real $\lambda_1 \neq \lambda_2$** ▸ $\begin{cases} \vec{x}_1(t) = e^{\lambda_1 t}\vec{v}_1 \\ \vec{x}_2(t) = e^{\lambda_2 t}\vec{v}_2 \end{cases}$ are both solutions to $\dot{\vec{x}} = A\vec{x}$

also need to check that $\vec{v}_1$ and $\vec{v}_2$ are linearly independent

↳ **General Solution** ▸ $\vec{X}(t) = C_1e^{\lambda_1 t}\vec{v}_1 + C_2e^{\lambda_2 t}\vec{v}_2$ $\quad C_1, C_2$ are defined by IC

Therefore $\{\vec{x}_1, \vec{x}_2\}$ are a fundamental set of solutions

The next step is to draw the general solution using a "phase portrait" using a change of base matrix P

Diagonalization

If A has a base of eigenvectors $\vec{v}_1, \vec{v}_2$, we can express it as $A = PDP^{-1}$, where $P = (\vec{v}_1, \vec{v}_2)$, $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \text{diag}(\lambda_1, \lambda_2)$

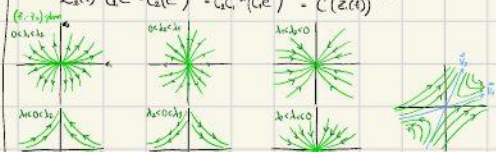
Let's make a change of variable $\vec{z} = P^{-1}\vec{x}$ $\Rightarrow \dot{\vec{z}} = P^{-1}\dot{\vec{x}} = P^{-1}A\vec{x} = P^{-1}PDP^{-1}\vec{x} = DP^{-1}\vec{x} = D\vec{z} \Rightarrow \dot{\vec{z}} = D\vec{z}$

Which does $\vec{z}$ solve?

so $\vec{z} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ satisfies $\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 z_1 \\ \lambda_2 z_2 \end{pmatrix} \Rightarrow \begin{cases} z_1(t) = C_1e^{\lambda_1 t} \\ z_2(t) = C_2e^{\lambda_2 t} \end{cases}$ Now graph the sol (z_1, z_2) obs the plane

$$\vec{z}_h(t) = C_1e^{\lambda_1 t} - C_2(e^{\lambda_1 t})^{\lambda_2/\lambda_1} = \tilde{C}(e^{\lambda_1 t})^{\lambda_2/\lambda_1}$$

$(\vec{v}_1, \vec{v}_2)$ plane



$$\vec{x} = P\vec{z} = (\vec{v}_1, \vec{v}_2)\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = C_1e^{\lambda_1 t}\vec{v}_1 + C_2e^{\lambda_2 t}\vec{v}_2$$

Ex. First find and draw phase diagram of $\begin{cases} \dot{x} = -x \\ \dot{y} = -3y \end{cases} \Rightarrow \vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}$, $A = \begin{pmatrix} -1 & 0 \\ 0 & -3 \end{pmatrix}$

$\dot{\vec{x}} = A\vec{x} \Rightarrow$ Eulers $\Rightarrow \begin{vmatrix} -1-\lambda & 0 \\ 0 & -3-\lambda \end{vmatrix} = (-1-\lambda)(-3-\lambda) = 0 \Rightarrow (\lambda+1)(\lambda+3) = \lambda^2 + 4\lambda + 3$

Eulers $\Rightarrow \lambda = \begin{pmatrix} a' + b' \\ a' + b' \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \end{pmatrix}$, $\begin{pmatrix} a' \\ b' \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\lambda = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ -3 \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\Rightarrow$ Gen Sol $\Rightarrow \vec{X}(t) = C_1e^{\lambda_1 t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2e^{\lambda_2 t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ ▸ $\lambda_2 = -3$, $\lambda_1 = -1$

Phase Diagram

in the variable $\vec{z} = P^{-1}\vec{x}$

- Since $\lambda_2 < 0 < \lambda_1$

To go back to $\vec{x}$ ▸ $\vec{x} = P\vec{z} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\vec{z}$



Repeated Case

if λ is $\lambda = 0$: $\vec{x}(t) = C_1\vec{v}e^{\lambda t}$

▷ Case 2. Complex Conjugate eigenvalues

Suppose A has an eigenvalue $\lambda = \rho \pm i\omega$ } Linear Algebra
then it has eigenvectors $\vec{v} = \vec{v}_1 \pm i\vec{v}_2$ }

$$\left[\frac{-b}{2a} = \pm\sqrt{\frac{b^2}{4a^2} - \frac{k}{a}}\, i\right]$$

– Then A has a complex diagonalization: $A = PDP^{-1}$ where $P = (\vec{v}_1\ \vec{v}_2)$
and $D = \begin{pmatrix} \rho+i\omega & 0 \\ 0 & \rho-i\omega \end{pmatrix}$

Let's rewrite as follows: $\left[\vec{v}_1 = \frac{\vec{v}_+ + \vec{v}_-}{2}\right]$ and $\left[\vec{v}_2 = -i\left(\frac{\vec{v}_+ - \vec{v}_-}{2}\right)\right]$

We define the matrix $Q = (\vec{v}_1\ \vec{v}_2) = P\begin{pmatrix} 1/2 & -i/2 \\ 1/2 & i/2 \end{pmatrix} = (\vec{v}_1\ \vec{v}_2)\begin{pmatrix} 1/2 & -i/2 \\ 1/2 & i/2 \end{pmatrix}$ and $P = Q\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}$

Define a variable: $\vec{z} = Q^{-1}\vec{x}$ (where $Q = (\vec{v}_1\ \vec{v}_2)$, $\vec{v}_i = (\vec{v}_i \in \mathbb{R})$)

What does $\vec{z}$ satisfy? : $\vec{z}' = Q^{-1}\vec{x}' = Q^{-1}A\vec{x} = Q^{-1}PDP^{-1}\vec{x} = \underline{Q^{-1}PDQ^{-1}}\vec{x} = \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}\begin{pmatrix} \rho-i\omega & 0 \\ 0 & \rho+i\omega \end{pmatrix}\begin{pmatrix} 1/2 & -i/2 \\ 1/2 & i/2 \end{pmatrix}\vec{z}$

$= \frac{1}{2}\begin{pmatrix} \rho-i\omega & \rho-i\omega \\ i\rho+\omega & -i\rho+\omega \end{pmatrix}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}\vec{z} = \frac{1}{2}\begin{pmatrix} 2\rho & 2\omega \\ 2\omega & 2\rho \end{pmatrix} = \begin{pmatrix} \rho & \omega \\ -\omega & \rho \end{pmatrix}$. If solve, we get a spiral counterclockwise for $\vec{z}$. $\begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$
$[z_1' z_2' - z_2' z_1'] \{0 - \text{counterclockwise}\}$

General Solution = $\vec{X}(t) = C_1 \mathrm{Re}\{\vec{v}e^{\lambda t}\} + C_2\, \mathrm{Im}\{\vec{v}e^{\lambda t}\}$

Particular Solution: $\vec{X}(t) = r(0)e^{\rho t}\left(\cos(\theta(0) - \omega t)\vec{v}_1 + \sin(\theta(0) - \omega t)\vec{v}_2\right)$

Example: solve $\dot{\vec{x}} = A\vec{x}$ for $A = \begin{pmatrix} 2 & 5 \\ -2 & 0 \end{pmatrix}$

Step 1: Find λ values: $\det(A - \lambda I) = 0 \triangleright (2-\lambda)(-\lambda) + 10 = 0 \triangleright \lambda^2 - 2\lambda + 10 = 0 \triangleright \lambda = \frac{2 \pm \sqrt{36}}{2} = 1 \pm 3i$

Step 2: Find λ vectors: $\left(\begin{array}{cc|c} 1-3i & 5 & 0 \\ -2 & -1-3i & 0 \end{array}\right) \triangleright \left(\begin{array}{cc|c} 1-3i & 5 & 0 \\ 1-3i & 5 & 0 \end{array}\right) \triangleright \left(\begin{array}{cc|c} 1-3i & 5 & 0 \\ 0 & 0 & 0 \end{array}\right)$, $\vec{v}_+ = \begin{pmatrix} -5 \\ 1-3i \end{pmatrix} = \underbrace{\begin{pmatrix} -5 \\ 1 \end{pmatrix}}_{\vec{v}_1} + i\underbrace{\begin{pmatrix} 0 \\ -3 \end{pmatrix}}_{\vec{v}_2}$

Step 3: Particular Solution: $\vec{X}(t) = r(0)e^{\rho t}\left(\cos(\theta(0) - 3t)\begin{pmatrix} -5 \\ 1 \end{pmatrix} + \sin(\theta(0) - 3t)\begin{pmatrix} 0 \\ -3 \end{pmatrix}\right)$

Step 4: Phase Plane Diagram: $\triangleright$ let $\vec{z}$, in the variable $z = Q^{-1}\vec{x}$, where $Q = (\vec{v}_1, \vec{v}_2)$; $\vec{z}$ satisfies $\vec{z}' = \begin{pmatrix} 1 & 5 \\ -1 & 1 \end{pmatrix}\vec{z}$, where solution is $\triangleright \vec{z}(t) = r(0)e^{t}\begin{pmatrix} \cos(\theta_0 - 3t) \\ \sin(\theta_0 - 3t) \end{pmatrix}$

– clockwise solution
→ "Unstable spiral" (since $\rho > 0$, $\omega > 0$)



Change coordinates $\triangleright$ to x_1, x_2 axis $\begin{pmatrix} 2 & 5 \\ -2 & 0 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$



• Still unstable spiral
• direction of rotation: go back to original system → $\begin{cases} \text{if } \dot{z}_2 > 0 & \text{clockwise} \\ \text{if } \dot{z}_2 < 0 & \text{counterclockwise} \end{cases}$
$\triangleright \dot{z}_2 = 5z_1 - 3z_2$? ?

▷ Case 3. Repeated Real Eigenvalue

Suppose A has only one eigenvalue.

SUBCASE 3.1. 2 different eigenvectors $\{\vec{v}_1, \vec{v}_2\}$

▷ We can diagonalize A as $A = PDP^{-1}$, where $P = (\vec{v}_1\ \vec{v}_2)$ and $D = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \lambda I$

▷ $A = PDP^{-1} \triangleright AP\vec{v} = \lambda P\vec{v} \triangleright (A - \lambda I)\vec{v} = 0 \triangleright \overset{\text{Alg.2}}{\vec{0}}\vec{v} = \vec{0}$, $\forall \vec{v} \in \mathbb{R}^2$.

▷ $\vec{V}$ is an eigenvector for all vectors $\Rightarrow \vec{X}(t) = \vec{X}_0 e^{\lambda t}$ is a particular solution for all $\vec{X}_0$

Phase diagrams



"Unstable Star" ($\lambda > 0$)



"Stable Star" ($\lambda < 0$)

SUBCASE 3.2. only one eigenvector $\{\vec{V}\}$ (we can't diagonalize A)

▷ choose $\vec{v}_2$ linindep $\vec{V}$, and define $\vec{v}_1 = (A - \lambda I)\vec{v}_2$, then, since $\{\vec{v}, \vec{v}_2\}$ is a basis for $\mathbb{R}^2$, we can write $\vec{v}_1 = \alpha\vec{v}_2 + \beta\vec{v}$

▷ Let's calculate $(A - \lambda I)\vec{v}_1 = (A - \lambda I)(\alpha\vec{v}_2 + \beta\vec{v}) = \alpha(A - \lambda I)\vec{v}_2 = \alpha\vec{v}_1 \triangleright \left(A - (\alpha + \lambda)I\right)\vec{v}_1 = 0$

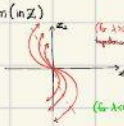
▷ $\vec{v}_1$ is an eigenvector for $(\lambda + \alpha)$, but since the only eigenvalue is $\lambda \triangleright \alpha = 0$, so we know $\vec{v}_1$ is graphic $\vec{v} \triangleright \{\vec{v}_1, \vec{v}_2\}$ basis

▷ Then $(A - \lambda I)\vec{v}_1 = \vec{v}_1 \triangleright A\vec{v}_2 = \vec{v}_1 + \lambda\vec{v}_2 \triangleright$ Let $P = (\vec{v}_1, \vec{v}_2) \triangleright AP = A(\vec{v}_1\ \vec{v}_2) = (\lambda\vec{v}_1\ \vec{v}_1 + \lambda\vec{v}_2) = (\vec{v}_1\ \vec{v}_2)\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \triangleright A = P\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}P^{-1}$

▷ Define $\vec{z} = P^{-1}\vec{x} \Rightarrow \dot{\vec{z}} = P^{-1}\dot{\vec{x}} = P^{-1}A\vec{x} = P^{-1}P\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}P^{-1}\vec{x} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}\vec{z}$ $\begin{cases} \dot{z}_1 = \lambda z_1 + z_2 \triangleright \dot{z}_1 - \lambda z_1 = Ce^{\lambda t} \triangleright \frac{\dot{z}_1 e^{-\lambda t} - \lambda z_1 e^{-\lambda t}}{} = C \triangleright (z_1 e^{-\lambda t})' = C \triangleright z_1(t) = (Ct + d)e^{\lambda t} \\ \dot{z}_2 = \lambda z_2 \longrightarrow z_2 = Ce^{\lambda t} \end{cases}$

General Solution ▷ $\vec{X} = P\vec{z} = (\vec{v}_1\ \vec{v}_2)\vec{z} = (Ct + D)e^{\lambda t}\vec{v}_1 + Ce^{\lambda t}\vec{v}_2$

Phase diagram (in $\vec{z}$)
• find the line where $\dot{\vec{z}} = 0$
• $\dot{z}_2 = -\lambda z$



Phase diagram (in $\vec{x}$)
• same as in $\vec{z}$, but trajectories are drawn towards $\vec{v}_1$
• in order to determine the shape, we need an eigenvalue A

