uc3m | Universidad Carlos III de Madrid

Degree in Aerospace Engineering

Advanced Mathematics

Second midterm, 2025-2026

Name:

Group: **Date: 05/12/2025**

Write in pen (not pencil) and explain your answers with the theory seen in class.

Problem 3 (0.5+2+0.5 points) We consider the following system of ODEs for $x(t)$ and $y(t)$ (derivatives are with respect to t):

$$\begin{aligned} x' &= y, \\ y' &= x^2 - 2x - y. \end{aligned}$$

1. Find the stationary points.
2. Linearise the system and classify the stationary points in the linear system.
3. Sketch the phase portrait of the original nonlinear system.

Solution:

1. If $x'(t) = y'(t) = 0$ at a stationary point, we obtain $y = 0$ from the first ODE, and replacing this in the second one we have $x = 0$ and $x = 2$. The stationary points are then $(0, 0)$ and $(2, 0)$.

2. The right hand side of the system is

$$\mathbf{F}(x, y) = \begin{pmatrix} y \\ x^2 - 2x - y \end{pmatrix} \Rightarrow \mathbf{JF}(x, y) = \begin{pmatrix} 0 & 1 \\ 2x - 2 & -1 \end{pmatrix}.$$

We linearise the system around the two stationary points, using this Jacobian. We start with $(0, 0)$, then

$$\mathbf{JF}(0, 0) = \begin{pmatrix} 0 & 1 \\ -2 & -1 \end{pmatrix}.$$

The eigenvalues of this matrix are given by $0 = -\lambda(-1 - \lambda) + 2 = \lambda^2 + \lambda + 2$, so $\lambda_\pm = -\frac{1}{2} \pm \mathrm{i}\frac{\sqrt{7}}{2}$, and this is a stable spiral point in the linear system (because the real part of the eigenvalues is < 0). We get the direction of rotation by taking (for example) a point such that $x(t) = 0$ and $y(t) > 0$. Then, from the first ODE, we obtain $x'(t) = y(t) > 0$, so the trajectories rotate clockwise..

Next

$$\mathbf{JF}(2, 0) = \begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix}.$$

The eigenvalues of this matrix are given by $0 = -\lambda(-1-\lambda) + 2 = \lambda^2 + \lambda - 2$, so $\lambda_1 = 2$ and $\lambda_2 = -1$, and this is a saddle point in the linear system. The eigenvectors are

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} 1 & -1 \end{pmatrix},$$

so this is an unstable spiral point (because the real part of the eigenvalues is > 0). We get the direction of rotation by taking (for example) a point such that $\xi(t) = 0$ and $\eta(t) > 0$. Then, from the first ODE, we obtain $\xi'(t) = -2\eta(t) < 0$, so the trajectories rotate counterclockwise.

3. Both stationary points remain of the same type in the nonlinear system, and the phase portrait looks like in Figure 1.

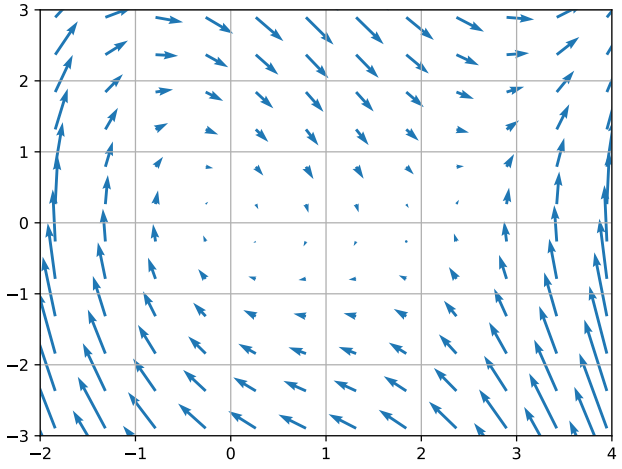


Figure 1: Phase portrait for Exercise 1.

Problem 2 (2+2 points) We study the following boundary problem:

$$y''(x) + \lambda y(x) = 0, \qquad y(-1) = y(1) = 0,$$

with $\lambda \geq 0$.

1. Calculate the eigenvalues λ_n and the corresponding eigenfunctions $y_n(x)$.
2. For $m, n \geq 0$, calculate

$$\int_{-1}^{1} y_n(x) y_m(x)\, \mathrm{d}x.$$

The following formulas may be useful:

$$\begin{aligned} \sin(\alpha \pm \beta) &= \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta), \\ \cos(\alpha \pm \beta) &= \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta) \\ 2\sin^2(\alpha) &= 1 - \cos(2\alpha), \qquad 2\cos^2(\alpha) = 1 + \cos(2\alpha) \end{aligned}$$

Solution:

1. We consider the cases $\lambda = 0$ and $\lambda > 0$ separately, because they lead to different solutions:

   - If $\lambda = 0$, we have $y(x) = Ax + B$, and the boundary conditions give $0 = -A + B$ (at $x = -1$) and $0 = A + B$ (at $x = 1$) . Then, it follows that $A = B = 0$ (adding and subtracting the two equations), so we only have the trivial solution, and $\lambda = 0$ is not an eigenvalue.
   - If $\lambda > 0$, then the general solution is

     $$y(x) = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x).$$

     If we impose $y(-1) = 0$ and $y(1) = 0$, we obtain

     $$\begin{aligned} c_1 \cos(\sqrt{\lambda}) + c_2 \sin(\sqrt{\lambda}) &= 0, \\ c_1 \cos(\sqrt{\lambda}) - c_2 \sin(\sqrt{\lambda}) &= 0 \end{aligned}$$

     If we add both equations, we obtain $c_1 \cos(\sqrt{\lambda}) = 0$, so either $c_1 = 0$ or $\cos(\sqrt{\lambda}) = 0$. In the first case, $y(x) = c_2 \sin(\sqrt{\lambda}x)$, and then the condition $y(1) = 0$ gives either $c_2 = 0$ (and then the solution is trivial), or $\sin(\sqrt{\lambda}) = 0$, in which case the eigenvalues are $\lambda_n = n^2\pi^2$, $n = 1, 2, \ldots$, with eigenfunctions $y_n(x) = \sin(n\pi x)$.
     If $\cos(\sqrt{\lambda}) = 0$, we have eigenvalues $\lambda_n = \left(n + \frac{1}{2}\right)^2 \pi^2$, $n = 0, 1, 2, \ldots$. Then the condition $y(1) = 0$ gives $c_2 = 0$, because $\sin(\sqrt{\lambda}) \neq 0$, and the eigenfunctions are $y_n(x) = \cos\left(\left(n + \frac{1}{2}\right)\pi x\right)$.
     A similar result is obtained if, for example, we subtract both equations instead of adding them. We can also start from the equality $c_1 \cos(\sqrt{\lambda}) + c_2 \sin(\sqrt{\lambda}) = c_1 \cos(\sqrt{\lambda}) - c_2 \sin(\sqrt{\lambda})$.

2. In the first case, with $n \neq m$, we have

   $$\begin{aligned} \int_{-1}^{1} y_n(x) y_m(x)\,\mathrm{d}x &= \int_{-1}^{1} \sin(n\pi x)\sin(m\pi x)\,\mathrm{d}x \\ &= \frac{1}{2}\left(\int_{-1}^{1} \cos((n-m)\pi x)\,\mathrm{d}x - \int_{-1}^{1} \cos((n+m)\pi x)\,\mathrm{d}x\right) \\ &= \frac{1}{2}\left(\frac{\sin((n-m)\pi x)}{(n-m)\pi}\Big|_{x=-1}^{x=1} - \frac{\sin((n+m)\pi x)}{(n+m)\pi}\Big|_{x=-1}^{x=1}\right) = 0. \end{aligned}$$

   If $m = n$, then

   $$\int_{-1}^{1} y_n^2(x)\,\mathrm{d}x = \int_{-1}^{1} \sin^2(n\pi x)\,\mathrm{d}x = \frac{1}{2}\int_{-1}^{1} (1 - \cos(2n\pi x))\,\mathrm{d}x = 1.$$

   In the second case, we have

   $$\begin{aligned} \int_{-1}^{1} y_n(x) y_m(x)\,\mathrm{d}x &= \int_{-1}^{1} \cos\left(\left(n + \tfrac{1}{2}\right)\pi x\right)\cos\left(\left(m + \tfrac{1}{2}\right)\pi x\right)\,\mathrm{d}x \\ &= \frac{1}{2}\left(\int_{-1}^{1} \cos((n+m+1)\pi x)\,\mathrm{d}x + \int_{-1}^{1} \cos((n-m)\pi x)\,\mathrm{d}x\right) \end{aligned}$$

If $m \neq n$ (and both ≥ 0), both integrals are 0:

$$\int_{-1}^{1} y_n(x) y_m(x)\,\mathrm{d}x = \frac{1}{2}\left(\frac{\sin\left((n+m+1)\pi x\right)}{(n+m+1)\pi}\Big|_{x=-1}^{x=1} + \frac{\sin\left((n-m)\pi x\right)}{(n-m)\pi}\Big|_{x=-1}^{x=1} \right) = 0.$$

If $m = n \neq 0$, then we have

$$\int_{-1}^{1} y_n^2(x)\,\mathrm{d}x = \int_{-1}^{1} \cos^2\left(\left(n+\tfrac{1}{2}\right)\pi x\right)\,\mathrm{d}x = \frac{1}{2}\int_{-1}^{1} \left(1+\cos\left((2n+1)\pi x\right)\,\mathrm{d}x\right) = 1.$$

Finally, if $m = n = 0$, we obtain

$$\int_{-1}^{1} y_0^2(x)\,\mathrm{d}x = \int_{-1}^{1} \cos^2\left(\frac{\pi x}{2}\right)\,\mathrm{d}x = \frac{1}{2}\int_{-1}^{1} \left(1+\cos\left(\pi x\right)\right)\,\mathrm{d}x = 1.$$

If we take one eigenfunction from each family, then

$$\int_{-1}^{1} y_n(x) y_m(x)\,\mathrm{d}x = \int_{-1}^{1} \sin\left(n\pi x\right)\cos\left(\left(m+\tfrac{1}{2}\right)\pi x\right)\,\mathrm{d}x = 0,$$

for any n and m, because the integrand is odd.

Problem 2 (2+1 points)

1. Calculate the Fourier series for the function:

$$f(x) = \begin{cases} 0, & -1 \leq x \leq 0, \\ x, & 0 \leq x \leq 1, \end{cases}$$

2. Use this Fourier series to calculate the value of the following sum:

$$S = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots$$

Solution:

1. We have to expand the function with sines and cosines of period $2L = 2$, so

$$f(x) \sim \sum_{n=0}^{\infty} a_n \cos\left(n\pi x\right) + \sum_{n=1}^{\infty} b_n \sin\left(n\pi x\right).$$

Then,

$$a_0 = \frac{\displaystyle\int_{-1}^{1} f(x)dx}{\displaystyle\int_{-1}^{1} dx} = \frac{1}{2}\int_{0}^{1} x dx = \frac{1}{4}$$

$$a_n = \frac{\displaystyle\int_{-1}^{1} f(x)\cos(n\pi x)dx}{\displaystyle\int_{-1}^{1} \cos^2(n\pi x)dx} = \int_{0}^{1} x\cos(n\pi x)dx = \frac{(-1)^n - 1}{n^2\pi^2},$$

$$b_n = \frac{\displaystyle\int_{-1}^{1} f(x)\sin(n\pi x)dx}{\displaystyle\int_{-1}^{1} \sin^2(n\pi x)dx} = \int_{0}^{1} x\sin(n\pi x)dx = \frac{(-1)^{n+1}}{n\pi},$$

for $n = 1, 2, 3, \ldots$

So, the Fourier series is

$$f(x) \sim \frac{1}{4} + \sum_{s=1}^{\infty} \frac{(-1)^n - 1}{n^2\pi^2} \cos(n\pi x) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\pi} \sin(n\pi x).$$

2. If we evaluate the Fourier series and the function at 0 (where the series is convergent), then the sine part of the series is 0, and in the cosine part only odd terms are nonzero, so we obtain

$$0 = \frac{1}{4} - 2\sum_{s=1}^{\infty} \frac{1}{(2s-1)^2\pi^2} = \frac{1}{4} - \frac{2}{\pi^2}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\right) = \frac{1}{4} - \frac{2}{\pi^2}S,$$

and as a consequence, $S = \dfrac{\pi^2}{8}$.
