EDOs: second midterm

Name: Group:

General Comments: Read carefully to answer all the questions. Frame the final answer in a nice box. The easier for me, the happier I will be marking. Check the solutions whenever possible. If asked to explain something in your own words, **a few sentences** are enough. Separation of variables is a must! Not knowing how to do it might penalize you. Remember that you can assume the solution of $X'' + \lambda X = 0$ with D.B.C. and N.B.C. Good luck!

1. Consider the heat equation on a one-dimensional rod of length L with non-homogeneous boundary conditions.

$$(\star) \quad \begin{cases} u_t(t,x) = u_{xx}(t,x) & x \in [0,L] \;\; t > 0 \\ u_x(t,0) = -2L & t > 0 \\ u_x(t,L) = 0 & t > 0 \\ u(0,x) = f(x) \end{cases}$$

(3) (a) Consider the energy of a solution u to $(\star)$, which is given by

$$E(t) = \int_{x=0}^{L} u(t,x)\mathrm{d}x$$

Taking the time-derivative of the previous expression, prove that $E(t)$ is not constant.

(5) (b) One way to solve this PDE with non-homogeneous B.C. is to consider the change of variable

$$v(t,x) = u(t,x) + a(x^2 + 2tb) + bx$$

Find $a, b \in \mathbb{R}$ such that v satisfies the homogeneous heat equation:

$$(\star\star) \quad \begin{cases} v_t(t,x) = v_{xx}(t,x) & x \in [0,L] \;\; t > 0 \\ v_x(t,0) = 0 & t > 0 \\ v_x(t,L) = 0 & t > 0 \end{cases}$$

(3) (c) Find the initial condition $g(x)$ of $(\star\star)$ in terms of the initial condition $f(x)$ of $(\star)$.

(15) (d) Using separation of variables, find the general solution to $(\star\star)$ (with no initial condition).

(5) (e) Find the particular solution that satisfies $v(0,x) = g(x)$ for a generic g.

(3) (f) Take the limit $t \to \infty$ of the solution v to the homogeneous problem $(\star\star)$ that you found in (d) and show that the system reaches a final equilibrium state regardless of the initial condition. Explain the physical meaning of that result.

(9) (g) Find the particular solution for u (careful, solve for v and then find u, the solution to the original problem $(\star)$) in the following cases:

$$i)\ g(x) = 3 - \cos(3\pi x/L) \qquad ii)\ g(x) = 2 + \sin(2\pi x/L) \qquad iii)\ g(x) = 0$$

2. Consider the nonlinear coupled differential equation

$$\begin{cases} \dot{x} = x(2 - x - y) \\ \dot{y} = y(6 - 6x - y) \end{cases}$$

(5) (a) Assuming this is an ecological model with two species, indicate if they are preys or predators. Indicate the effect that one population has over the other.

(5) (b) Find the stationary points of the system.

(12) (c) Draw the linearized behaviour for each equilibrium. For that, use the following facts

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \longrightarrow \begin{matrix} \lambda_1 = a & v_1 = (1,0) \\ \lambda_2 = d & v_2 = (b, d-a) \end{matrix} \qquad \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \longrightarrow \begin{matrix} \lambda_1 = a & v_1 = (a-d, c) \\ \lambda_2 = d & v_2 = (0,1) \end{matrix}$$

and assume that the equilibrium (x_0, y_0) for $x_0, y_0 > 0$ is a saddle with eigenvectors $v_+ = (-0.6, 1)$ (the one associated to the positive eigenvalue) and $v_- = (0.4, 1)$.

(12) (d) Draw, to the best of your abilities (make it big please and with colors if possible), the full phase diagram.

3. A particle of mass $m > 0$ orbiting a much heavier object of mass $M > 0$ (located at the origin) is characterized by two conserved quantities: its energy $E \in \mathbb{R}$ and its angular momentum $L \in \mathbb{R}$. In terms of the polar coordinates $(r(t), \theta(t))$ of m, they are given by (we assume the orbit is contained in the xy plane):

$$E = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - G\frac{mM}{r} \qquad\qquad L = mr^2\dot{\theta}$$

Where $G > 0$ is the gravitational constant. Removing the angular velocity $\dot{\theta}$ we reach:

$$E = \frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2} - G\frac{mM}{r}$$

$V_{\text{eff}}(r) = \dfrac{L^2}{2mr^2} - G\dfrac{mM}{r}$ is called the effective potential that we take as our potential.

(8) (a) Assuming $L \neq 0$, draw a sketch of $V_{\text{eff}}(r)$ for $r > 0$ (use its limits and its derivative).

(8) (b) Explain **briefly** the effect of reducing the value of L. What happens with the graph if $L = 0$? What is the physical meaning of that (check the definition of L)?

(6) (c) Draw the phase diagram in the $(r, \dot{r})$ plane ($L \neq 0$). Consider at least four energies: the minimum energy, an energy between the minimum and zero, the zero energy and a positive energy.

(6) (d) Sketch the four types of orbits obtained in (c) in the xy plane (draw the four possible orbits of the particle orbiting M) explaining **briefly** their correspondence with the previous diagram.