ADVANCED MATHEMATICS

Bachelor Degree in Aerospace Engineering, Universidad Carlos III de Madrid

Group 45, course 2021-2022

December 10th, 2021

Total duration: 1.5 hours

Note: In order to get full marks, you have to **justify** the calculations and **explain** (briefly) the method that you use!

Name:

P1 (0.5+3+0.5 points) We consider the following PDE for $u(x,t)$:

$$\begin{aligned} u_t &= u_{xx} - u, && x \in (0,1), \quad t > 0, \\ u_x(0,t) &= u_x(1,t) = 0, && t > 0, \\ u(x,0) &= f(x), && x \in (0,1). \end{aligned}$$

1. Calculate the stationary solution $v(x)$ (independent of time) for this problem.
2. Solve the problem using separation of variables, with $f(x) = \sin^4(\pi x)$.
3. Does the stationary solution satisfy $v(x) = \lim_{t\to\infty} u(x,t)$?

Solution:

1. If $v(x)$ is a stationary solution, it satisfies

$$\begin{aligned} 0 &= v'' - v, && x \in (0,1), \\ v'(0) &= v'(1) = 0. \end{aligned}$$

So we have $v(x) = c_1 e^x + c_2 e^{-x}$. If $v'(0) = 0$, we get $c_1 - c_2 = 0$, and if $v'(1) = 0$, then $c_1 e - c_2 e^{-1} = 2c_1 \sinh 1 = 0$, so $v(x) = 0$.

2. We look for solutions of the form $u(x,t) = X(x)T(t)$, and then the heat equation becomes $XT' = X''T - XT$, then dividing by XT and rearranging, we obtain

$$\frac{T'}{T} + 1 = \frac{X''}{X} = -\lambda.$$

This leads to two ODEs, the spatial one is

$$X'' + \lambda X = 0, \qquad X'(0) = X'(1) = 0.$$

- If $\lambda < 0$, then $X(x) = c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x}$. Now, $X'(0) = 0$ gives $c_1 - c_2 = 0$, and $X'(1) = 0$ leads to

$$c_1 \sqrt{-\lambda}(e^{\sqrt{-\lambda}} + e^{-\sqrt{-\lambda}}) = 0.$$

If $c_1 = 0$ we obtain the trivial solution, and otherwise we need $1 + e^{-2\sqrt{-\lambda}} = 0$, so $\lambda = 0$. It follows that there are no negative eigenvalues.

- If $\lambda = 0$, then $X(x) = Ax + B$. Now, $X'(0) = 0$ gives $A = 0$, and $X'(1) = 0$ leads to $A = 0$ again, so $\lambda = 0$ is an eigenvalue with eigenfunction $X_0(x) = B$, a constant (not necessarily 0).
- If $\lambda > 0$, then $X(x) = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x)$. In this case, $X'(0) = 0$ gives $c_2 = 0$, and $X'(1) = 0$ leads to

$$c_1 \sin(\sqrt{\lambda}) = 0.$$

  If $c_1 = 0$, we have the trivial solution. If $c_1 \neq 0$, we have eigenvalues $\lambda_n = n^2\pi^2$, $n = 1, 2, \ldots$, with eigenfunctions $X_n(x) = \cos(n\pi x)$.

Then, the time ODE is $T' + (1 + \lambda)T = 0$, so $T' + (1 + n^2\pi^2)T = 0$, therefore $T(t) = Ce^{-(1+n^2\pi^2)t}$. The principle of superposition gives

$$u(x,t) = \sum_{n=0}^{\infty} a_n e^{-(1+n^2\pi^2)t} \cos(n\pi x).$$

We want $u(x, 0) = f(x)$, so

$$f(x) = \sum_{n=0}^{\infty} a_n \cos(n\pi x),$$

and then a_n are the Fourier cosine coefficients of the function $f(x)$, given by

$$a_n = 2\int_0^1 f(x)\cos(n\pi x)dx, \qquad n = 1, 2, \ldots, \qquad a_0 = \int_0^1 f(x)dx.$$

If $f(x) = \sin^4(\pi x)$, we can write

$$f(x) = \frac{(1 - \cos(2\pi x))^2}{4} = \frac{1}{4} - \frac{\cos(2\pi x)}{2} + \frac{1 + \cos(4\pi x)}{8}.$$

Therefore, we can identify coefficients

$$a_0 = \frac{3}{8}, \qquad a_2 = -\frac{1}{2}, \qquad a_4 = \frac{1}{8},$$

and $a_n = 0$ otherwise. Then the solution is

$$u(x,t) = \frac{3}{8}e^{-t} - \frac{1}{2}e^{-(1+4\pi^2)t}\cos(2\pi x) + \frac{1}{8}e^{-(1+16\pi^2)t}\cos(4\pi x).$$

3. All the exponential terms in t are negative, so $\lim_{t\to\infty} u(x,t) = 0$, which coincides with the stationary solution.

P2t (1+1+1 points) We consider the following ODE:

$$y''(x) - 2xy'(x) + \lambda y(x) = 0, \qquad y'(0) = y'(1) = 0.$$

1. Write it as a Sturm–Liouville problem and identify the coefficients $p(x)$, $q(x)$ and $\sigma(x)$.

2. Use Rayleigh's quotient to show that the eigenvalues of this problem satisfy $\lambda \geq 0$. Is $\lambda = 0$ an eigenvalue? If so, what it its eigenfunction?

3. What orthogonality property do the eigenfunctions satisfy?

Solution:

1. If we want

$$y''(x) - 2xy'(x) = (p(x)y'(x))' = p'(x)y'(x) + p(x)y''(x)$$

we don't find any solution for $p(x)$, so we try to add an integrating factor $I(x)$. Then

$$I(x)y''(x) - 2xI(x)y'(x) = (p(x)y'(x))' = p'(x)y'(x) + p(x)y''(x),$$

so $p(x) = I(x)$ and $p'(x) = -2xI(x)$. Differentiating the first equation, we obtain $p'(x) = I'(x)$, therefore $I'(x) = -2xI(x)$, which means $I(x) = e^{-x^2}$, and then $p(x) = e^{-x^2}$, and the ODE becomes

$$(e^{-x^2}y'(x))' + e^{-x^2}\lambda y(x) = 0.$$

It follows that $q(x) = 0$ and $\sigma(x) = e^{-x^2}$.

2. The Rayleigh quotient in this case gives

$$\lambda = \frac{-p(x)y(x)y'(x)\Big|_{x=0}^{x=1} + \displaystyle\int_0^1 [p(x)(y'(x))^2 - q(x)y(x)^2]dx}{\displaystyle\int_0^1 y(x)^2\sigma(x)dx} = \frac{0 + \displaystyle\int_0^1 e^{-x^2}(y'(x))^2dx}{\displaystyle\int_0^1 y(x)^2e^{-x^2}dx}.$$

This ratio is ≥ 0. If $\lambda = 0$, then we have $y'(x) = 0$, so $y(x)$ is constant, but the information from the endpoints is $y'(0) = y'(1) = 0$, so $\lambda = 0$ is an eigenvalue, with a constant eigenfunction.

3. The eigenfunctions $\phi_n(x)$ satisfy the orthogonality property

$$\int_0^1 \phi_n(x)\phi_m(x)\sigma(x)dx = \int_0^1 \phi_n(x)\phi_m(x)e^{-x^2}dx = 0, \qquad n \neq m.$$

P3 (1+2 points) We consider a particle of mass $m = 1$Kg that moves on the real line subject to the potential

$$V(x) = \frac{x^5}{5} - x.$$

The position of the particle, as function of time, is $x(t)$.

1. Impose that the total energy

$$E(t) = \frac{1}{2}\dot{x}^2 + V(x)$$

is constant and deduce a second order ODE for $x(t)$. Write this ODE as a 2×2 system.

2. Find stationary points of the previous system, linealise it, classify the stationary points and sketch the phase portrait.

Solution:

1. We calculate

$$\frac{dE}{dt} = 2\dot{x}\ddot{x} + V'(x)\dot{x}.$$

If this is equal to 0 and $\dot{x} \neq 0$, we obtain the second order ODE

$$\ddot{x} = -V'(x) = -x^4 + 1.$$

As a 2×2 system, if $v = \dot{x}$, we obtain:

$$\begin{aligned} \dot{x} &= v, \\ \dot{v} &= -x^4 + 1 \end{aligned}$$

2. We have two stationary points at $(x, v) = (\pm 1, 0)$. The Jacobian is

$$J(x, v) = \begin{pmatrix} 0 & 1 \\ -4x^3 & 0 \end{pmatrix} \Rightarrow J(\pm 1, 0) = \begin{pmatrix} 0 & 1 \\ \mp 4 & 0 \end{pmatrix}$$

At $x = -1$, the eigenvalues of this matrix are $\lambda = \pm 2$, and the eigenvectors are

$$v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

The origin (in the linear system) is a saddle point, and the same is true for the nonlinear system.

At $x = 1$, the eigenvalues of this matrix are $\lambda = \pm 2i$, so the origin is a centre.

If we take the energy, we have

$$E(x, v) = \frac{1}{2}v^2 + \frac{x^5}{5} - x,$$

with stationary points at $(\pm 1, 0)$. Since the Hessian is

$$H(x, v) = \begin{pmatrix} 12x^3 & 0 \\ 0 & 1 \end{pmatrix},$$

and then positive definite at $x = 1$, $(1, 0)$ is a local minimum, so trajectories close to that point are closed, and the point remains a centre.

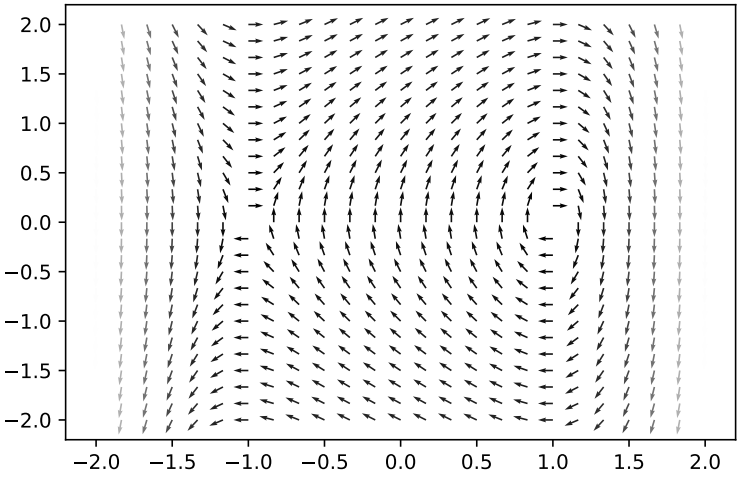


Figure 1: Phase portrait