ADVANCED MATHEMATICS

Bachelor Degree in Aerospace Engineering, Universidad Carlos III de Madrid

Group 47, course 2020-2021

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Total duration: 2 hours

PROBLEM 1. (5 points)

We consider Laplace's equation on a wedge, in polar coordinates (r, θ):

$$\begin{cases} \Delta u = u_{rr} + \dfrac{1}{r} u_r + \dfrac{1}{r^2} u_{\theta\theta} = 0, \quad 0 < r < 1, \quad 0 < \theta < \dfrac{\pi}{4}, \\ u(r, 0) = u\left(r, \dfrac{\pi}{4}\right) = 0, \\ u(1, \theta) = f(\theta). \end{cases}$$

- Solve the problem by separation of variables, assuming that $u(r, \theta)$ is bounded as $r \to 0$. **Hint:** For the radial part look for solutions of the form $R(r) = r^\alpha$.
- Calculate the solution if $f(\theta) = \sin(4\theta) - \sin(12\theta)$.

Solution:

- We look for a solution in the form $u(r, \theta) = R(r)T(\theta)$, then we get

$$R''(r)T(\theta) + \frac{R'(r)T(\theta)}{r} + \frac{R(r)T''(\theta)}{r^2} = 0 \Rightarrow \frac{r^2 R''(r) + rR'(r)}{R(r)} = -\frac{T''(\theta)}{T(\theta)} = \lambda.$$

  Then for the angular part we have $T'' + \lambda T = 0$, with boundary conditions $T(0) = T\left(\frac{\pi}{4}\right) = 0$.

  - If $\lambda < 0$, we have $T(\theta) = Ae^{\sqrt{-\lambda}\theta} + Be^{-\sqrt{-\lambda}\theta}$, and the boundary conditions give $A + B = 0$ and $Ae^{\sqrt{-\lambda}\frac{\pi}{4}} + Be^{-\sqrt{-\lambda}\frac{\pi}{4}}$. Since $A = -B$, we have $2A \sinh \sqrt{-\lambda}\frac{\pi}{4} = 0$; therefore $A = 0$, then $B = 0$ and we have the trivial solution.
  - If $\lambda = 0$, then $T(\theta) = A\theta + B$, and the boundary conditions give $B = 0$ and then $A\frac{\pi}{4} = 0$, so trivial solution again.
  - If $\lambda > 0$, we obtain $T(\theta) = A\cos\left(\sqrt{\lambda}\theta\right) + B\sin\left(\sqrt{\lambda}\theta\right)$. If $T(0) = 0$ then $A = 0$, and then $T\left(\frac{\pi}{4}\right)$ gives
  $$B\sin\left(\sqrt{\lambda}\,\frac{\pi}{4}\right) = 0 \Rightarrow \lambda = 16n^2, \qquad n = 1, 2, \ldots,$$
  and then $T(\theta) = B\sin(4n\theta)$.

  For the radial part, we have $r^2R'' + rR' - \lambda R = 0$. If $R(r) = r^\alpha$, then we obtain $r^\alpha(\alpha(\alpha-1)+\alpha-\lambda) = 0$. If $r \neq 0$, then we obtain $\alpha = \pm\sqrt{\lambda} = \pm 4n$. Therefore,

$$R(r) = Ar^{4n} + Br^{-4n}.$$

  Since the solution is bounded as $r \to 0$, we have $B = 0$ and $R(r) = Ar^{4n}$. Therefore, we have the solution in the form of the following Fourier series:

$$u(r, \theta) = \sum_{n=1}^{\infty} a_n r^{4n} \sin(4n\theta).$$

If we impose $u(1,\theta) = f(\theta)$, we have

$$f(\theta) = \sum_{n=1}^{\infty} a_n \sin(4n\theta),$$

so

$$a_n = \frac{8}{\pi} \int_0^{\pi/2} f(\theta) \sin(4n\theta) d\theta,$$

are the Fourier sine coefficients of the boundary function $f(\theta)$.

- If $f(\theta) = \sin(4\theta) - \sin(12\theta)$, then we can identify the coefficients $a_1 = 1$, $a_3 = -1$ and $a_n = 0$ if $n \neq 1, 3$, therefore

$$u(r,\theta) = a_1 r^4 \sin(4\theta) + a_3 r^{12} \sin(12\theta) = r^4 \sin(4\theta) - r^{12} \sin(12\theta).$$

PROBLEM 2. (5 points)

Consider the following model of two population species:

$$x' = x(1 - x + ay)$$
$$y' = y(4 - 2y + bx),$$

1.1 Describe first the type of situation being modelled, in the cases $a, b > 0$ and $a, b < 0$.

1.2 Sketch the phase portrait for $x, y \geq 0$ when $a = -1$, $b = -5$.

1.3 Discuss what the phase portrait means for the evolution of the two species.

Solution:

- If $a, b > 0$ then it is a cooperative system, since an increase in any of the two populations will contribute to an increase in the rate of growth of the other (for example, larger y means larger x' in the first equation). If $a, b < 0$, it is a competitive system, since an increase in any of the two populations will contribute to a decrease in the rate of growth of the other (larger y means smaller x' in the first equation).

- In this case, we have

$$x' = x(1 - x - y)$$
$$y' = y(4 - 2y - 5x),$$

and we have the stationary points $(0,0)$, $(1,0)$, $(0,2)$ and $\left(\frac{2}{3}, \frac{1}{3}\right)$. The Jacobian matrix is

$$J(x,y) = \begin{pmatrix} 1 - 2x - y & -x \\ -5y & 4 - 4y - 5x \end{pmatrix},$$

so we have

$$J(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix},$$

with eigenvalues $\lambda = 1$ and $\lambda = 4$, so it is an unstable node. Next,

$$J(1,0) = \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix},$$

with eigenvalues $\lambda = -1$ (double), so it is an improper stable node. The eigenvector satisfies

$$\begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = - \begin{pmatrix} u \\ v \end{pmatrix} \Rightarrow \mathbf{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

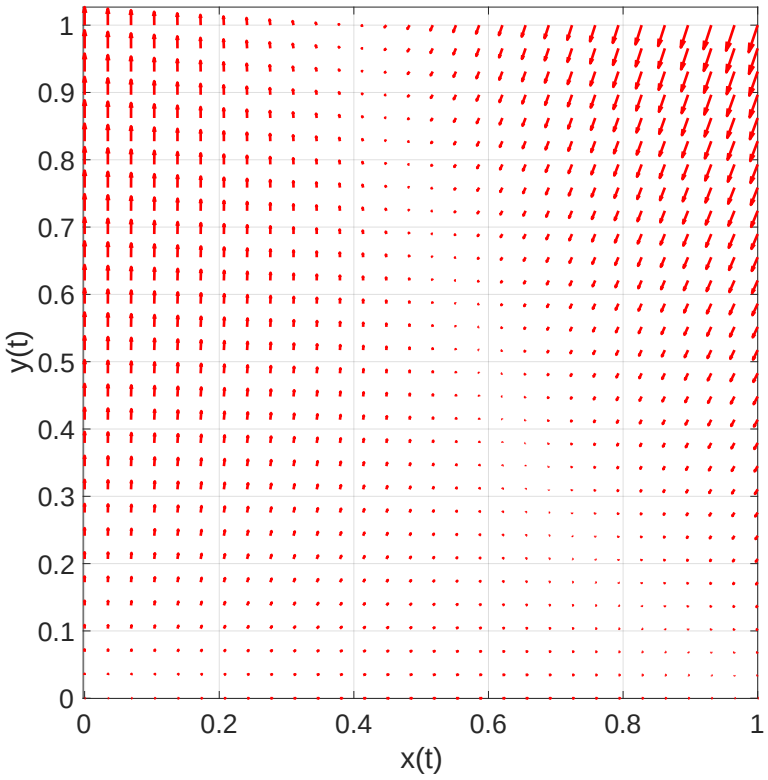


Figure 1: Phase portrait for Exercise 2.

Furthermore, if we consider $\xi = 0$ and $\eta > 0$, for example, then the linear system is $\dot{\xi} = -\xi - \eta = -\eta < 0$, and if $\dot{\eta} = 0$ then we have $\xi = -\eta$, so the trajectories cross this straight line vertically.

Next,

$$J(0,2) = \begin{pmatrix} -1 & 0 \\ -10 & -4 \end{pmatrix},$$

with eigenvalues $\lambda = -1$ and $\lambda = -4$, so it is a stable node. Finally

$$J\left(\frac{2}{3}, \frac{1}{3}\right) = \begin{pmatrix} -2/3 & -2/3 \\ -5/3 & -2/3 \end{pmatrix},$$

with eigenvalues $\lambda = \frac{-2\pm\sqrt{10}}{3}$, so it is a saddle point. The eigenvectors satisfy

$$\begin{pmatrix} -2/3 & -2/3 \\ -5/3 & -2/3 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \frac{-2+\sqrt{10}}{3} \begin{pmatrix} u \\ v \end{pmatrix} \Rightarrow \mathbf{v}_1 = \begin{pmatrix} 2 \\ -\sqrt{10} \end{pmatrix},$$
$$\begin{pmatrix} -2/3 & -2/3 \\ -5/3 & -2/3 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \frac{-2-\sqrt{10}}{3} \begin{pmatrix} u \\ v \end{pmatrix} \Rightarrow \mathbf{v}_2 = \begin{pmatrix} 2 \\ \sqrt{10} \end{pmatrix},$$

- The trajectories will tend to the stable points $(1,0)$ and $(0,2)$, which represent extinction of the second or first species (respectively), unless we start on the lines given by $\mathbf{v}_2$, which is the stable manifold for the saddle point $\left(\frac{2}{3}, \frac{1}{3}\right)$.