ADVANCED MATHEMATICS

Bachelor Degree in Aerospace Engineering, Universidad Carlos III de Madrid

Group 46, course 2020-2021

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Total duration: 2 hours

PROBLEM 1. (5 points)

Consider the following model of two population species:

$$\begin{aligned} x' &= x(1-x-y) \\ y' &= y\left(\frac{3}{4}-y-\frac{1}{2}x\right), \end{aligned}$$

1.1 Describe first the type of situation being modelled

1.2 Determine their stability type and sketch the phase portrait for $x, y \geq 0$.

1.3 Decide what the phase portrait means for the evolution of the two species.

Solution:

1.1 On its own x evolves according to $x' = x(1-x)$, a normal population equation. So x can survive on the natural resources available. Similarly, on its own y satisfies $y' = y\left(\frac{3}{4}-y\right)$, so y is also living on natural resources. The interaction terms means that y disadvantages x and vice-versa, so we have a competition situation.

Also, we observe that the coordinate axes are themselves trajectories. This follows directly since $x' = 0$ on the y-axis (where $x = 0$) and, similarly, $y' = 0$ on the x-axis. Thus no other trajectories can cross the coordinate axes. For a population problem, only non-negative values of x and y are significant, and we conclude that any trajectory that starts in the first quadrant must remain there for all time.

1.2 First we find the critical points by solving the system of algebraic equations

$$\begin{aligned} x(1-x-y) &= 0, \\ y\left(\frac{3}{4}-y-\frac{1}{2}x\right) &= 0. \end{aligned}$$

The first equation can be satisfied by choosing $x = 0$, then the second equation requires that $y = 0$ or $y = 3/4$. Similarly, the second equation can be satisfied by choosing $y = 0$, and then the first equation requires that $x = 0$ or $x = 1$. Thus we have found three critical points, namely, $(0, 0)$, $(0, 3/4)$, and $(1, 0)$. We have also the solution of the system

$$1 - x - y = 0, \quad 3/4 - y - 1/2x = 0,$$

which leads to a fourth critical point $(1/2, 1/2)$. The four critical points correspond to equilibrium solutions of the system.

The system is almost linear in the neighbourhood of each critical point. There are two ways to obtain the linear system near a critical point (x_0, y_0). First, we can use the substitution $x = x_0 + \xi$, $y = y_0 + \eta$ into

the system, retaining only the terms that are linear in ξ and η. Alternatively, we can evaluate the Jacobian matrix, providing us with the linearization of the system around a given point (x_0, y_0)

$$D\mathbf{F}(x_0, y_0) = \begin{pmatrix} 1 - 2x - y & -x \\ -\frac{1}{2}y & \frac{3}{4} - 2y - \frac{1}{2}x \end{pmatrix},$$

to find the coefficient matrix of the approximating linear system. When several critical points are to be investigated, it is usually more efficient to use the Jacobian matrix. By evaluating the Jacobian, we obtain the linear system

$$\frac{d}{dt}\begin{pmatrix}\xi\\ \eta\end{pmatrix} = \begin{pmatrix} 1 - 2x - y & -x \\ -\frac{1}{2}y & \frac{3}{4} - 2y - \frac{1}{2}x \end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix},$$

where $\xi = x - x_0$ and $\eta = y - y_0$. Such a system is the approximate linear system near the critical point (x_0, y_0). We will now examine each of the four critical points in turn.

At $(0, 0)$

$$D\mathbf{F}(0, 0) = \begin{pmatrix} 1 & 0 \\ 0 & 3/4 \end{pmatrix}$$

which has as eigenvalues $\lambda_1 = 1$, associated with the eigenvector $\mathbf{v}_1 = (1, 0)^T$ and $\lambda_2 = 3/4$ associated with $\mathbf{v}_2 = (0, 1)^T$ and its general solution

$$\begin{pmatrix}\xi\\ \eta\end{pmatrix} = c_1 e^t \begin{pmatrix}1\\ 0\end{pmatrix} + c_2 e^{3t/4}\begin{pmatrix}0\\ 1\end{pmatrix}, \quad \text{with} \quad c_1, c_2 \in \mathbb{R}.$$

Thus the origin is an unstable node of both the linear system and, thanks to the approximation, the nonlinear system. In the neighbourhood of the origin, all trajectories are tangent to the y-axis except for one trajectory that lies along the x-axis, since $|\lambda_2| = 3/4 < |\lambda_1| = 1$.

At $(1, 0)$, this corresponds to a state in which species x survives the competition but species y does not. By evaluating the Jacobian matrix for $x_0 = 1$, $y_0 = 0$, we find that the linear system corresponding to the system near the critical point $(1, 0)$ is

$$D\mathbf{F}(1, 0) = \begin{pmatrix} -1 & -1 \\ 0 & 1/4 \end{pmatrix}$$

which has as eigenvalues $\lambda_1 = -1$, associated with the eigenvector $\mathbf{v}_1 = (1, 0)^T$ and $\lambda_2 = 1/4$ associated with $\mathbf{v}_2 = (4, -5)^T$ and its general solution

$$\begin{pmatrix}\xi\\ \eta\end{pmatrix} = c_1 e^{-t} \begin{pmatrix}1\\ 0\end{pmatrix} + c_2 e^{t/2}\begin{pmatrix}4\\ -5\end{pmatrix}, \quad \text{with} \quad c_1, c_2 \in \mathbb{R}.$$

Since the eigenvalues have opposite signs, the point $(1, 0)$ is a saddle point, and hence is an unstable equilibrium point of the linear system and of the nonlinear system. The behaviour of the trajectories near $(1, 0)$ can be seen from the general solution. If $c_2 = 0$, then there is one pair of trajectories that approaches the critical point along the x-axis. If $c_1 = 0$, then another pair of trajectories leaves the critical point (more precisely, approaches it as $t \to -\infty$) tangent to the line of slope $-5/4$. All other trajectories also depart from the neighbourhood of $(1, 0)$.

At $(0, 3/4)$, species y survives but x does not. The analysis is similar to that for the point $(1, 0)$. The corresponding linear system is

$$D\mathbf{F}(0, 3/4) = \begin{pmatrix} 1/4 & -1 \\ -3/8 & -3/4 \end{pmatrix}$$

which has as eigenvalues $\lambda_1 = 1/4$, associated with the eigenvector $\mathbf{v}_1 = (8, -3)^T$ and $\lambda_2 = -3/4$ associated with $\mathbf{v}_2 = (0, 1)^T$ and its general solution

$$\begin{pmatrix}\xi\\ \eta\end{pmatrix} = c_1 e^{t/4} \begin{pmatrix}8\\ -3\end{pmatrix} + c_2 e^{-3t/4}\begin{pmatrix}0\\ 1\end{pmatrix}, \quad \text{with} \quad c_1, c_2 \in \mathbb{R}.$$

Thus the point $(0, 3/4)$ is also a saddle point. One pair of trajectories approaches the critical point along the y-axis and another departs tangent to the line with slope $-3/8$. All other trajectories also leave the neighbourhood of the critical point.

At $(1/2, 1/2)$. This critical point corresponds to a mixed equilibrium state, or coexistence, in the competition between the two species. The eigenvalues and eigenvectors of the corresponding linear system with the Jacobian matrix

$$D\mathbf{F}(1/2, 1/2) = \begin{pmatrix} -1/2 & -1/2 \\ -1/4 & -1/2 \end{pmatrix}.$$

which has as eigenvalues $\lambda_1 = \frac{-2+\sqrt{2}}{4} \sim -0.146$ associated with the eigenvector $\mathbf{v}_1 = (\sqrt{2}, -1)^T$ and $\lambda_2 = \frac{-2-\sqrt{2}}{4} \sim -0.854$ associated with the eigenvector $\mathbf{v}_1 = (\sqrt{2}, 1)^T$ and its general solution

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = c_1 e^{\lambda_1 t} \begin{pmatrix} \sqrt{2} \\ -1 \end{pmatrix} + c_2 e^{\lambda_2 t} \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix}, \quad \text{with} \quad c_1, c_2 \in \mathbb{R}.$$

Since both eigenvalues are negative, the critical point $(1/2, 1/2)$ is an asymptotically stable node of the linear system and of the nonlinear system. All trajectories approach the critical point as $t \to \infty$. One pair of trajectories approaches the critical point along the line with slope $\sqrt{2}/2$ determined from the eigenvector $\mathbf{v}_2$. All other trajectories approach the critical point tangent to the line with slope $-\sqrt{2}/2$ determined from the eigenvector $\mathbf{v}_1$.

1.3 A phase portrait for this system is shown in Figure . By looking closely at the trajectories near each critical point, you can see that the first three of these points involve the extinction of one or both species; only the last corresponds to the long-term survival of both species. In addition, note that the quadratic terms on the right side of the system are all negative. Since for x and y large and positive, these terms are the dominant ones, it follows that far from the origin in the first quadrant both x and y are negative, that is, the trajectories are directed inward. Thus all trajectories that start at a point (x_0, y_0) with $x_0 > 0$ and $y_0 > 0$ eventually approach the point $(1/2, 1/2)$.

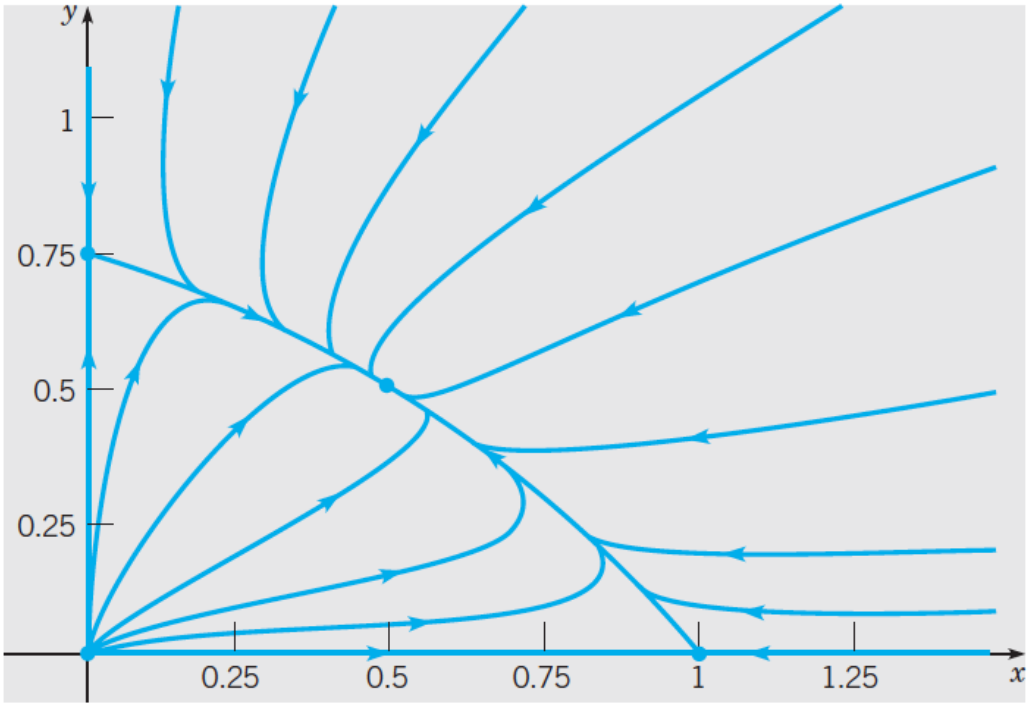


Figure 1: Phase portrait.

PROBLEM 2. (5 points)

Consider Laplace's equation on the rectangle $\{0 < x < L, 0 < y < H\}$

$$\begin{cases} \Delta u = 0 & 0 < x < L,\ 0 < y < H, \\ \partial_x u(0,y) = \partial_x u(L,y) = 0 & 0 < y < H, \\ \partial_y u(x,0) = 0,\ \partial_y u(x,H) = f(x) & 0 < x < L. \end{cases}$$

2.1 Obtain a so-called compatibility condition so that the problem may have a solution. Then use the the Divergence Theorem, i.e. for a domain D and a vector filed $\mathbf{F}$ it follows that $\int_D \mathrm{div}\mathbf{F} = \int_{\partial D} \mathbf{F}\cdot\mathbf{n}$, where $\mathbf{n}$ stands for the outward unit normal vector to ∂D, to justify such an answer.

2.2 Solve the problem by separation of variables. Prove that the method provides a solution only under the compatibility condition derived in part (i).

2.3 The solution obtained in (ii) depends on an arbitrary constant. Fix it by considering the relation of the problem with that of heat conduction on a rectangle with initial condition $u(x,y,0) = g(x,y)$.

Solution:

2.1 Thanks to the Divergence Theorem we find that

$$\int_\Omega \Delta u = \int_\Omega \mathrm{div}(\nabla u) = \int_{\partial\Omega} \nabla u\cdot\mathbf{n} = \int_{\partial\Omega} \frac{\partial u}{\partial \mathbf{n}},$$

where $\mathbf{n}$ stands for the outward unit normal vector to $\partial\Omega$ and ∇u is a vector field. Therefore, since $\int_\Omega \Delta u = 0$ it follows that $\int_{\partial\Omega} \frac{\partial u}{\partial \mathbf{n}} = 0$. In particular, we have that

$$\begin{aligned} \int_0^L\int_0^H \Delta u\,dx\,dy &= \int_0^L\int_0^H \frac{d^2u}{dx^2}\,dy\,dx + \int_0^L\int_0^H \frac{d^2u}{dy^2}\,dy\,dx \\ &= \int_0^H\int_0^L \frac{d^2u}{dx^2}\,dx\,dy + \int_0^L\int_0^H \frac{d^2u}{dy^2}\,dy\,dx \\ &= \int_0^H \frac{du}{dx}\Big]_0^L dy + \int_0^L \frac{du}{dy}\Big]_0^H dy\,dx \\ &= \int_0^H (\partial_x u(L,y) - \partial_x u(0,y))\ dy + \int_0^L (\partial_y u(x,H) - \partial_y u(x,0))\ dx \\ &= \int_0^L f(x)\,dx. \end{aligned}$$

The previous argument could be performed using the normal while applying directly the Divergence Theorem. Indeed, it is easy to see that the vector $(1,0)^T$ stands for the outward unit normal vector to the segment $\{x = L\}\times[0,H]$ of the rectangle $\{0 < x < L, 0 < y < H\}$, while $(-1,0)^T$ will be the outward unit normal vector to the segment $\{x = 0\}\times[0,H]$. On the other hand, for the segments $[0,L]\times\{y = H\}$ and $[0,L]\times\{y = 0\}$ we get the normal vector $(0,1)^T$ and $(0,-1)^T$ respectively. This leads to

$$\int_\Omega \Delta u = \int_\Omega \mathrm{div}(\nabla u) = \int_{\partial\Omega} \nabla u\cdot\mathbf{n} = \int_0^L \nabla u(x,H)\cdot\begin{pmatrix}0\\1\end{pmatrix} = \int_0^L f(x)\,dx = 0.$$

2.2 We now use the method of separation of variables to look for solutions of the Laplace equation, i.e. $u(x,y) = X(x)Y(y)$:

$$X''(x)Y(y) + X(x)Y''(y) = 0 \quad \Longrightarrow \quad \frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = -\lambda\,.$$

Thus, we find two ODEs:

$$\begin{cases} X''(x) + \lambda X(x) = 0, \\ Y''(y) - \lambda Y(x) = 0. \end{cases}$$

The boundary conditions become,

$$\begin{cases} \partial_x u(0,y) = X'(0)Y(y) = 0 \Rightarrow X'(0) = 0, \\ \partial_x u(L,y) = X'(0)Y(y) = 0 \Rightarrow X'(L) = 0, \\ \partial_y u(x,0) = X(x)Y'(0) = 0 \Rightarrow Y'(0) = 0. \end{cases}$$

Hence, first we solve the eigenvalue problem in x:

$$\begin{cases} X''(x) + \lambda X(x) = 0 \quad \Longrightarrow \quad X(x) = e^{rx}, \quad r^2 + \lambda = 0 \quad \Longrightarrow \quad r = \pm\sqrt{-\lambda}\,, \\ X'(0) = X'(L) = 0\,, \end{cases}$$

In particular,

- **For** $\lambda < 0 : r = \pm\sqrt{\lambda} \quad \Longrightarrow \quad X(x) = c_1 e^{(}\sqrt{-\lambda}\,x) + c_2 e^{(}-\sqrt{-\lambda}\,x)$, with

  $$X(x) = \sqrt{-\lambda}\left(c_1 e^{(}\sqrt{-\lambda}\,x) - c_2 e^{(}-\sqrt{-\lambda}\,x)\right).$$

  So that

  $$X'(0) = 0 \Rightarrow c_1 = c_2,$$

  and, hence,

  $$X'(L) = c_1\sqrt{-\lambda}\left(c_1 e^{(}\sqrt{-\lambda}\,L) - c_2 e^{(}-\sqrt{-\lambda}\,L)\right) = 0 \quad \Longrightarrow \quad c_1 = c_2 = 0\,.$$

  leading us to the trivial solution $X(x) \equiv 0$.
- **For** $\lambda = 0$: $r = 0 \quad \Longrightarrow \quad X(x) = c_1 + c_2 x$.

  $$X'(0) = c_1 = 0, \quad X'(L) = c_1 = 0 \quad \Longrightarrow \quad X_0(x) = 1.$$

- **for** $\lambda > 0 : r = \pm i\sqrt{\lambda} \quad \Longrightarrow \quad X(x) = c_1\cos(\sqrt{\lambda}\,x) + c_2\sin(\sqrt{\lambda}\,x)$.

  $$X'(0) = c_2 = 0, \quad X'(L) = c_1\sin(\sqrt{\lambda}\,L) = 0 \quad \Longrightarrow \quad \sqrt{\lambda}\,L = n\pi \Longrightarrow \lambda_n = \frac{n^2\pi^2}{L^2},\ n \in \mathbb{R},$$

  such that those eigenvalues are associated with the eigenvectors $X_n(x) = \cos\left(\frac{n\pi}{L}x\right)$, with $n \in \mathbb{N}$.

Next we solve the problem for the variable y and with $\lambda = \frac{n^2\pi^2}{L^2}$, such that $n \in \mathbb{N} \cup \{0\}$,

$$\left\{\ Y''(y) - \tfrac{n^2\pi^2}{L^2}Y(x) = 0 \quad \Longrightarrow \quad Y(x) = e^{ry}, \quad r^2 - \tfrac{n^2\pi^2}{L^2} = 0 \quad \Longrightarrow \quad r = \pm\tfrac{n\pi}{L}\,,\right.$$

Also, since $Y'(H) = 0$ and $Y'(y) = \frac{n\pi}{L}(c_1 e^{\frac{n\pi y}{L}} - c_2 e^{-\frac{n\pi y}{L}})$ it follows that

$$Y'(H) = 0 \Rightarrow \frac{n\pi}{L}(c_1 - c_2) = 0 \Rightarrow c_1 = c_2.$$

Therefore,

$$Y_n(y) = c_n(e^{\frac{n\pi y}{L}} + e^{-\frac{n\pi y}{L}}) = c_n\cosh\left(\frac{n\pi y}{L}\right), \quad \text{with} \quad n \in \mathbb{N} \cup \{0\}.$$

Applying the superposition principle to the family of solutions

$$u_n(x,y) = X_n(x)Y_n(y) = c_n \cos\left(\frac{n\pi}{L}x\right)\cosh\left(\frac{n\pi y}{L}\right), \quad \text{with} \quad n \in \mathbb{N}\cup\{0\},$$

we arrive at

$$u(x,y) = c_0 + \sum_{n=1}^{\infty} c_n \cos\left(\frac{n\pi}{L}x\right)\cosh\left(\frac{n\pi y}{L}\right).$$

Finally, we also have that $\partial_y u(x,H) = f(x)$ so that

$$f(x) = \sum_{n=1}^{\infty} c_n \cos\left(\frac{n\pi}{L}x\right)\frac{n\pi}{L}\cosh\left(\frac{n\pi H}{L}\right),$$

and due to the compatibility condition there will be solutions

$$\text{for any } c_0 \in \mathbb{R} \text{ and such that} \quad \int_0^L f(x)\,dx = 0.$$

2.3 By definition of stationary solution $u(x,y) = \lim_{t\to\infty} v(x,y,t)$ where $v(x,y,t)$ is the solution of the parabolic equation

$$\begin{cases} \frac{\partial v}{\partial t} = \Delta v & 0<x<L,\ 0<y<H,\ t>0, \\ \partial_x v(0,y,t) = \partial_x v(L,y,t) = 0 & 0<y<H, \\ \partial_y v(x,0,t) = 0,\ \partial_y v(x,H,t) = f(x) & 0<x<L, \\ v(x,y,0) = g(x,y) & 0<x<L,\ 0<y<H. \end{cases}$$

The energy is given by

$$E(t) = \int_R v \Rightarrow E'(t) = \int_R \frac{\partial v}{\partial t} = \int_R \Delta v = \int_{\partial R} \frac{\partial v}{\partial \mathbf{n}} = \int_0^L f(x)\,dx = 0.$$

Hence, we deduce the conservation of energy from the previously obtained compatibility condition. Consequently, we find that

$$\int_0^L \int_0^H g(x,y)\,dx\,dy = \int_0^L \int_0^H v(x,y,0)\,dx\,dy = c_0 LH,$$

and, c_0 will be the mean value of the initial data in the rectangle R. Note that all the higher order terms vanish at the limit.