ADVANCED MATHEMATICS

Bachelor Degree in Aerospace Engineering, Universidad Carlos III de Madrid

Group 45, course 2020-2021

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Total duration: 2 hours

PROBLEM 1. (5 points)

Consider the following model of two population species:

$$x' = x(2 - x + ay)$$
$$y' = y(3 - y + bx),$$

1.1 Describe first the type of situation being modelled, in the cases $a, b > 0$ and $a, b < 0$.

1.2 Sketch the phase portrait for $x, y \geq 0$ when $a = -1$, $b = -2$.

1.3 Discuss what the phase portrait means for the evolution of the two species.

Solution:

- If $a, b > 0$ then it is a cooperative system, since an increase in any of the two populations will contribute to an increase in the rate of growth of the other (for example, larger y means larger x' in the first equation). If $a, b < 0$, it is a competitive system, since an increase in any of the two populations will contribute to a decrease in the rate of growth of the other (larger y means smaller x' in the first equation).

- In this case, we have

$$x' = x(2 - x - y)$$
$$y' = y(3 - y - 2x),$$

and we have the stationary points $(0, 0)$, $(2, 0)$, $(0, 3)$ and $(1, 1)$. The Jacobian matrix is

$$J(x, y) = \begin{pmatrix} 2 - 2x - y & -x \\ -2y & 3 - 2y - 2x \end{pmatrix},$$

so we have

$$J(0, 0) = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix},$$

with eigenvalues $\lambda = 2$ and $\lambda = 3$, so it is an unstable node, with eigenvectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Next,

$$J(2, 0) = \begin{pmatrix} -2 & -2 \\ 0 & -1 \end{pmatrix},$$

with eigenvalues $\lambda = -1$ and $= -2$, so it is a stable node, with eigenvectors

$$\mathbf{v}_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

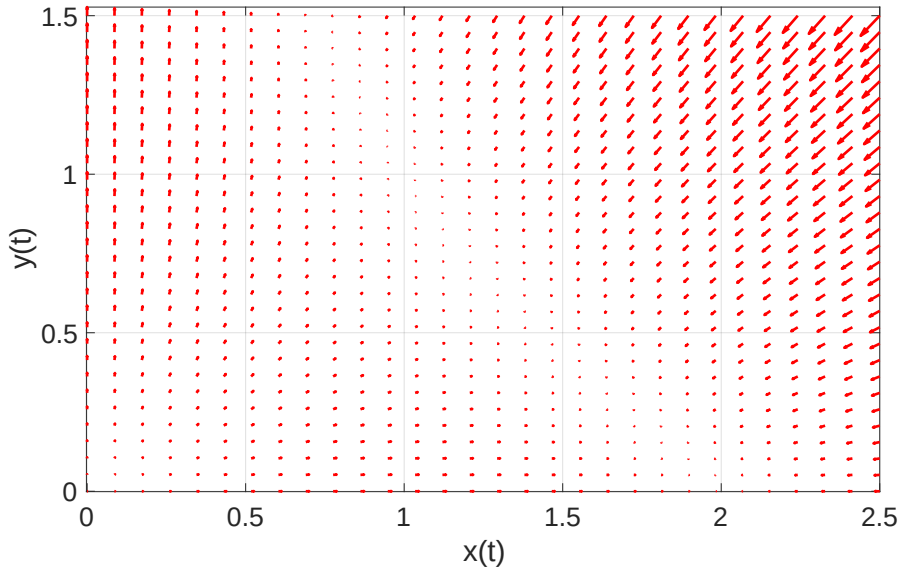


Figure 1: Phase portrait for Exercise 1.

Next,

$$J(0,3) = \begin{pmatrix} -1 & 0 \\ -6 & -3 \end{pmatrix},$$

with eigenvalues $\lambda = -1$ and $\lambda = -3$, so it is a stable node, with eigenvectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Finally

$$J(1,1) = \begin{pmatrix} -1 & -1 \\ -2 & -1 \end{pmatrix},$$

with eigenvalues $\lambda = -1 \pm \sqrt{2}$, so it is a saddle point. The eigenvectors satisfy

$$\begin{pmatrix} -1 & -1 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = (-1+\sqrt{2}) \begin{pmatrix} u \\ v \end{pmatrix} \Rightarrow \mathbf{v}_1 = \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix},$$
$$\begin{pmatrix} -1 & -1 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = (-1-\sqrt{2}) \begin{pmatrix} u \\ v \end{pmatrix} \Rightarrow \mathbf{v}_2 = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}.$$

- The trajectories will tend to the stable points $(2,0)$ and $(0,3)$, which represent extinction of the second or first species (respectively), unless we start on the line given by $\mathbf{v}_2$, which is the stable manifold for the saddle point $(1,1)$.

PROBLEM 2. (5 points)

We consider Laplace's equation in a half disc, in polar coordinates (r,θ):

$$\begin{cases} \Delta u = u_{rr} + \dfrac{1}{r} u_r + \dfrac{1}{r^2} u_{\theta\theta} = 0, \quad 0 < r < 1, \quad 0 < \theta < \pi, \\ u(r,0) = u(r,\pi) = 0, \\ u(1,\theta) = f(\theta). \end{cases}$$

- Solve the problem by separation of variables, assuming that $u(r,\theta)$ is bounded as $r \to 0$. **Hint:** For the radial part look for solutions of the form $R(r) = r^\alpha$.
- Calculate the solution if $f(\theta) = \sin(\theta) + 2\sin(2\theta)$.

Solution:

- We look for a solution in the form $u(r,\theta) = R(r)T(\theta)$, then we get

$$R''(r)T(\theta) + \frac{R'(r)T(\theta)}{r} + \frac{R(r)T''(\theta)}{r^2} = 0 \Rightarrow \frac{r^2R''(r) + rR'(r)}{R(r)} = -\frac{T''(\theta)}{T(\theta)} = \lambda.$$

Then for the angular part we have $T' + -\lambda T = 0$, with boundary conditions $T(0) = T(\pi) = 0$.

  - If $\lambda < 0$, we have $T(\theta) = Ae^{\sqrt{-\lambda}\theta} + Be^{-\sqrt{-\lambda}\theta}$, and the boundary conditions give $A + B = 0$ and $Ae^{\sqrt{-\lambda}\pi} + Be^{-\sqrt{-\lambda}\pi}$. Since $A = -B$, we have $2A\sinh\sqrt{-\lambda}\pi = 0$; therefore $A = 0$, then $B = 0$ and we have the trivial solution.
  - If $\lambda = 0$, then $T(\theta) = A\theta + B$, and the boundary conditions give $B = 0$ and then $A\pi = 0$, so we find the trivial solution again.
  - If $\lambda > 0$, we obtain $T(\theta) = A\cos\left(\sqrt{\lambda}\theta\right) + B\sin\left(\sqrt{\lambda}\theta\right)$. If $T(0) = 0$ then $A = 0$, and then $T(\pi)$ gives
$$B\sin\left(\sqrt{\lambda}\pi\right) = 0 \Rightarrow \lambda = n^2, \qquad n = 1, 2, \ldots,$$
and then $T(\theta) = B\sin(n\theta)$.

For the radial part, we have $r^2R'' + rR' - \lambda R = 0$. If $R(r) = r^\alpha$, then we obtain $r^\alpha(\alpha(\alpha-1)+\alpha-\lambda) = 0$. If $r \neq 0$, then we obtain $\alpha = \pm\sqrt{\lambda} = \pm n$. Therefore,

$$R(r) = Ar^n + Br^{-n}, \qquad n = 1, 2, \ldots$$

Since the solution is bounded as $r \to 0$, we have $B = 0$ and then $R(r) = Ar^n$. Therefore, we have the solution in the form of the following Fourier series:

$$u(r,\theta) = \sum_{n=1}^{\infty} a_n r^n \sin(n\theta).$$

If we impose $u(1,\theta) = f(\theta)$, we have

$$f(\theta) = \sum_{n=1}^{\infty} a_n \sin(n\theta),$$

so

$$a_n = \frac{2}{\pi}\int_0^\pi f(\theta)\sin(n\theta)d\theta,$$

are the Fourier sine coefficients of the boundary function $f(\theta)$.

- If $f(\theta) = \sin(\theta) + 2\sin(2\theta)$, then we can identify the coefficients $a_1 = 1$, $a_2 = 2$ and $a_n = 0$ if $n \neq 1, 2$, therefore

$$u(r,\theta) = a_1 r\sin(\theta) + a_2 r^2\sin(12\theta) = r\sin(\theta) + 2r^2\sin(2\theta).$$
