PROBLEM 1.

Let

$$f(x) = \begin{cases} \frac{5x}{2} & \text{if} \quad 0 < x < \frac{2}{3}, \\ 3 - 2x & \text{if} \quad \frac{2}{3} < x < 1. \end{cases}$$

1.1 Find the cosine Fourier series for the function $f(x)$.

1.2 Sketch the periodic function corresponding to the Fourier cosine series in the whole real line.

SOLUTION:

We are looking for the cosine series in the interval $[0, 1]$, hence,

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos(n\pi x),$$

such that the coefficients are obtained after using the even extension $\hat{f}(x)$. Thus, we compute the Fourier cosine series for this even extension

$$\hat{f}(x) = \hat{A}_0 + \sum_{n=1}^{\infty} \hat{A}_n \cos(n\pi x),$$

such that

$$\hat{A}_n = \int_{-1}^{1} \hat{f}(x) \cos(n\pi x) dx = 2 \int_0^1 f(x) \cos(n\pi x) dx = A_n, \quad n = 0, 1, 2, \ldots$$

For $n = 0$ we have that

$$A_0 = 2 \int_0^1 f(x) dx = \frac{5x^2}{2} \Big]_0^{2/3} + (6x - 2x^2) \Big]_{2/3}^1 = 2.$$

Integrating rest of the terms

$$\begin{aligned} A_n &= 2 \int_0^1 f(x) \cos(n\pi x) dx = 5 \int_0^{2/3} x \cos(n\pi x) dx + 2 \int_{2/3}^1 (3 - 2x) \cos(n\pi x) dx \\ &= 5 \frac{x \sin(n\pi x)}{n\pi} \Big]_0^{2/3} - \frac{5}{n\pi} \int_0^{2/3} \sin(n\pi x) dx + \frac{6}{n\pi} \sin(n\pi x) \Big]_{2/3}^1 \\ &- 4 \frac{x \sin(n\pi x)}{n\pi} \Big]_{2/3}^1 + \frac{4}{n\pi} \int_{2/3}^1 \sin(n\pi x) dx \\ &= \frac{9}{n^2\pi^2} \cos\left(\frac{2n\pi}{3}\right) - \frac{5}{n^2\pi^2} - \frac{4(-1)^n}{n^2\pi^2} \end{aligned}$$

PROBLEM 2

2.1) The eigenvalue problem for θ is of the form

$$\phi'' + \lambda \phi = 0$$

with periodic boundary conditions $\phi(-\pi) = \phi(\pi)$, $\phi'(-\pi) = \phi'(\pi)$, hence there is a zero-eigenvalue. The solution of the problem will read then

$$u(r,\theta) = u_0(r) + \sum_{n=1}^{\infty} \left(A_n u_n(r) \cos(n\theta) + B_n u_n(r) \sin(n\theta) \right)$$

because $\phi_0(\theta) = 1$. Since the boundary condition at $r = 2$ only involves $\sin 3\theta$, the term $u_0(r)$ has to be identically zero, as with terms involving any $\cos n\theta$ and $\sin n\theta$ for $n \neq 3$.

2.2) Assume $u(r,\theta) = \rho(r)\phi(\theta)$; then

$$\nabla^2 u = 0 = \frac{1}{r}\rho'\phi + \rho\phi'' + \frac{1}{r^2}\rho\phi'' = 0$$

$$\Rightarrow \underbrace{\frac{r^2\rho'' + r\rho'}{\rho}}_{\text{dep. } r} = \underbrace{-\frac{\phi''}{\phi}}_{\text{dep. } \theta} = \lambda$$

$$\begin{cases} \phi'' + \lambda\phi = 0 \\ \phi(-\pi) = \phi(\pi) \\ \phi'(-\pi) = \phi'(\pi) \end{cases} \quad (\star)$$

$$r^2\rho'' + r\rho' = \lambda\rho \quad (\star\star)$$

The solution of $(\star)$ is

$$\lambda_n = \frac{n^2\pi^2}{\pi^2} = n^2, \quad n = 0, 1, 2, \ldots$$

$$\phi_0(\theta) = 1, \quad \phi_n(\theta) = \begin{cases} \sin(n\theta) \\ \cos(n\theta) \end{cases} \quad n > 0$$

Then, considering $(\star\star)$ with $n \neq 0$ (due to 2.1):

$$\left.\begin{aligned} r^2\rho'' + r\rho' &= n^2\rho \\ \rho(r) &= r^\alpha \end{aligned}\right\} \quad \alpha(\alpha - 1) + \alpha = n^2 \Rightarrow \alpha^2 = n^2$$

$$\alpha = \pm n$$

Hence,

$$\left.\begin{array}{l} \rho_n(r) = A_n r^n + \dfrac{B_n}{r^n} \\ \rho_n(1) = 0 \\ v(1,\theta) = \rho(1)\phi(\theta) = 0 \end{array}\right\} \quad B_n = -A_n \Rightarrow \rho_n(r) = A_n\left(r^n - \frac{1}{r^n}\right)$$

The general solution reads (using 2.1)

$$v(r,\theta) = A_3\left(r^3 - \frac{1}{r^3}\right)\sin(3\theta)$$

Using

$$v(2,\theta) = \sin(3\theta)$$

$$A_3\left(8 - \frac{1}{8}\right) = 1$$

$$\Downarrow$$

$$A_3 = \frac{8}{63}$$

and

$$\boxed{v(r,\theta) = \frac{8}{63}\left(r^3 - \frac{1}{r^3}\right)\sin(3\theta)}$$