uc3m | Universidad Carlos III de Madrid

Degree in Aerospace Engineering

Advanced Mathematics

First midterm, 2025-2026

Name:

Group: **Date: 23/10/2025**

- No electronic devices, books, or notes can be used in the exam.
- In all exercises, **explain your answers with the theory seen in class**.

Problem 3 (1+2 points) Given the following IVP

$$y''(x) + y(x) = 0, \quad y(0) = 1, \quad y'(0) = 1.$$

1. Solve it using the general theory of linear second order ODEs.
2. Calculate a power series solution at $x_0 = 0$ and identify the series in terms of elementary functions.

Solution:

1. The general solution will be $y(x) = y_h(x) + y_p(x)$, where $y_h(x)$ is the general solution of the homogeneous ODE, and $y_p(x)$ a particular solution of the inhomogeneous ODE. In this case, if we set $y_h(x) = e^{\lambda x}$, we obtain from the homogeneous ODE $e^{\lambda x}(\lambda^2 + 1) = 0$. Since $e^{\lambda x} \neq 0$, we solve $\lambda^2 + 1 = 0$, which gives $\lambda = \pm i$, so we can take
$$y_h(x) = C_1 e^{ix} + C_2 e^{-ix}.$$
This is a pair of independent solutions (we can check the Wronskian). As al alternative, we can take the real solutions
$$y_h(x) = \widetilde{C}_1 \cos(x) + \widetilde{C}_2 \sin(x),$$
which are independent as well. The initial conditions give $y(0) = 1 = C_1$ and $y'(0) = 1 = C_2$, so $C_1 = 1$ and $C_2 = 1$, and then
$$y(x) = \cos(x) + \sin(x).$$

2. Since the coefficients of the ODE can be written as power series at $x = x_0$, we propose a solution of the form
$$y(x) = \sum_{n=0}^{\infty} a_n x^n,$$
and then the ODE becomes
$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} x^n = 0,$$

and we find the recursive relation

$$a_{n+2} = -\frac{a_n}{(n+2)(n+1)}, \qquad n \geq 0.$$

The initial values give $a_0 = y(0) = 1$ and $a_1 = y'(0) = 1$, so

$$a_2 = -\frac{1}{2!}, \qquad a_4 = \frac{1}{4!}, \qquad a_6 = -\frac{1}{6!}, \qquad \dots$$

for the even coefficients, and

$$a_3 = -\frac{1}{3!}, \qquad a_5 = \frac{1}{5!}, \qquad a_7 = -\frac{1}{7!}, \qquad \dots$$

for the odd ones. The resulting series is

$$y(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = \cos(x) + \sin(x).$$

identifying the Taylor series of sine and cosine.

Problem 2 (2 points) Solve the following ODE:

$$2x + 3y + (3x + 2y)\frac{dy}{dx} = 0.$$

Calculate the particular solution with initial value $y(0) = 2$.

Solution: We write this ODE as $f(x, y) + g(x, y)\frac{dy}{dx} = 0$, with coefficients

$$f(x, y) = 2x + 3y, \qquad g(x, y) = 3x + 2y.$$

Since $f_y = 3 = g_x$, this equation is exact. The general solution is given by $F(x, y) = C$, where C is a constant and $F(x, y)$ is such that $F_x = f$ and $F_y = g$. The first equality gives $F_x = 2x + 3y$, so $F(x, y) = x^2 + 3xy + h(y)$, for some function $h(y)$. Then the second equality gives $F_y = 3x + h'(y) = 3x + 2y$, so $h(y) = y^2$. We obtain

$$x^2 + 3xy + y^2 = C.$$

If $y(0) = 2$, then we have $0 + 0 + 4 = C$, so

$$x^2 + 3xy + y^2 = 4.$$

Problem 3 (5 points) Given the following system of differential equations

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t), \qquad A = \begin{pmatrix} a & 1 \\ 1 & a \end{pmatrix}, \qquad \mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix},$$

where $a \in \mathbb{R}$ is a constant, calculate the general solution using eigenvalues/eigenvectors. For which values of a is the origin a stable node, an unstable node and a saddle point? Calculate the particular solution with $a = 0$ and initial values $\mathbf{x}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Solution: We calculate the eigenvalues and eigenvectors of the matrix:

$$0 = \det(A - \lambda I) = \det \begin{pmatrix} a-\lambda & 1 \\ 1 & a-\lambda \end{pmatrix} = (a-\lambda)^2 - 1 = \lambda^2 - 2a\lambda + a^2 - 1,$$

so the eigenvalues are

$$\lambda_\pm = \frac{2a \pm \sqrt{4a^2 - 4a^2 + 4}}{2},$$

so $\lambda_1 = a + 1$ and $\lambda_2 = a - 1$. Note that these eigenvalues are real and different for any a, so we can diagonalise the system, using the eigenvectors

$$A\mathbf{v}_1 = (a+1)\mathbf{v}_1 \Rightarrow \begin{pmatrix} a & 1 \\ 1 & a \end{pmatrix} \mathbf{v}_1 = (a+1)\mathbf{v}_1 \Rightarrow \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$A\mathbf{v}_2 = (a-1)\mathbf{v}_2 \Rightarrow \begin{pmatrix} a & 1 \\ 1 & a \end{pmatrix} \mathbf{v}_2 = (a-1)\mathbf{v}_2 \Rightarrow \mathbf{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

In the new variables given by

$$\mathbf{x}(t) = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \mathbf{z}(t)$$

we have the decoupled system

$$\mathbf{z}'(t) = \begin{pmatrix} a+1 & 0 \\ 0 & a-1 \end{pmatrix} \mathbf{z}(t).$$

Therefore, since eigenvales are different, the general solution is

$$\mathbf{z}(t) = C_1 e^{(a+1)t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{(a-1)t} \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

and in the original variables

$$\mathbf{x}(t) = C_1 e^{(a+1)t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{(a-1)t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

If $a < -1$, then both eigenvalues are negative, and the origin is a stable node. If $-1 < a < 1$, then we have eigenvalues of different sign, so the origin is a saddle point. If $a > 1$, then both eigenvalues are positive, so the origin is an unstable node.

If $a = 0$, we have

$$\mathbf{x}(t) = C_1 e^{t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{-t} \begin{pmatrix} -1 \\ 1 \end{pmatrix},$$

and the initial values $\mathbf{x}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ give

$$C_1 - C_2 = 0, \qquad C_1 + C_2 = 1,$$

so $C_1 = C_2 = \frac{1}{2}$, and we obtain the particular solution

$$\mathbf{x}(t) = \begin{pmatrix} \frac{1}{2}(e^t - e^{-t}) \\ \frac{1}{2}(e^t + e^{-t}) \end{pmatrix}.$$
