uc3m | Universidad Carlos III de Madrid

Degree in Aerospace Engineering

Advanced Mathematics

First midterm, 2025-2026

Name:

Group: **Date: 24/10/2025**

- No electronic devices, books, or notes can be used in the exam.
- In all exercises, **explain your answers with the theory seen in class**.

Problem 1 (1+2 points) We consider the following IVP:

$$y'(x) = y(x)^2 + 2y(x) + 1, \qquad y(0) = 1.$$

(a) Solve it as a separable ODE.

(b) Calculate the largest interval $(-a, a)$ around the origin in which the solution is defined.

(c) Write $y(x) = -\frac{\varphi'(x)}{\varphi(x)}$, for some function $\varphi(x)$ which is $\neq 0$ in some interval around $x = 0$. Use the ODE for $y(x)$ to obtain a second order linear ODE for $\varphi(x)$ and solve it. Check that you obtain the same result as before!

Solution:

(a) If we separate variables, we have

$$\frac{y'(x)}{y(x)^2 + 2y(x) + 1} = 1,$$

so integrating in x, we obtain

$$\int \frac{y'(x)}{y(x)^2 + 2y(x) + 1} dx = \int dx \Rightarrow -\frac{1}{y(x) + 1} = x + C,$$

using the fact that $y(x)^2 + 2y(x) + 1 = (y(x) + 1)^2$. Then, the general solution is

$$y(x) = -1 - \frac{1}{x + C}.$$

If we impose now $y(0) = 1$, we obtain $C = -\frac{1}{2}$, so the particular solution becomes

$$y(x) = -1 - \frac{2}{2x - 1} = \frac{1 + 2x}{1 - 2x}.$$

(b) The solution is well defined as long as the denominator does not vanish, which happens at $x^* = \frac{1}{2}$. Therefore, the solution is defined in $\left(-\frac{1}{2}, \frac{1}{2}\right)$, so $a = \frac{1}{2}$.

(b) If we write the ODE in terms of $\varphi(x)$, we obtain

$$-\frac{\varphi''(x)}{\varphi(x)} + \left(\frac{\varphi'(x)}{\varphi(x)}\right)^2 = \left(\frac{\varphi'(x)}{\varphi(x)}\right)^2 + 2\frac{\varphi'(x)}{\varphi(x)} + 1 \Rightarrow -\frac{\varphi''(x)}{\varphi(x)} = +2\frac{\varphi'(x)}{\varphi(x)} + 1,$$

which leads to

$$\varphi''(x) - 2\varphi'(x) + \varphi(x) = 0,$$

which is a second order linear homogeneous ODE with constant coefficients. If we set $\varphi(x) = e^{\lambda x}$, we obtain $e^{\lambda x}(\lambda^2 - 2\lambda + 1) = 0$. Since $e^{\lambda x} \neq 0$, we solve $(\lambda^2 - 2\lambda + 1) = (\lambda - 1)^2 = 0$, and the general solution is

$$\varphi(x) = (c_1 + c_2 x)e^x.$$

It follows that

$$y(x) = -\frac{\varphi'(x)}{\varphi(x)} = -\frac{(c_2 + c_1 + c_2 x)e^x}{(c_1 + c_2 x)e^x} = -\frac{c_2 + c_1 + c_2 x}{c_1 + c_2 x}.$$

Since $y(0) = 1 = -\frac{\varphi'(0)}{\varphi(0)}$, so $\varphi'(0) = -\varphi(0)$, which gives $c_1 = -c_1 - c_2$, and then $c_1 = -\frac{1}{2}c_2$. Then,

$$y(x) = -\frac{\varphi'(x)}{\varphi(x)} = -\frac{\frac{c_2}{2} + c_2 x}{-\frac{c_2}{2} + c_2 x} = -\frac{1 + 2x}{-1 + 2x} = \frac{1 + 2x}{1 - 2x},$$

which is the same as before. We can simplify c_2 above, because if $c_2 = 0$, then $c_1 = 0$ and it follows that $\varphi(x) = 0$, which contradicts the assumption on $\varphi(x)$ in the exercise.

Problem 2 (4 points) Given the following system of differential equations

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t), \qquad A = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix}, \qquad \mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix},$$

calculate the general solution using eigenvalues/eigenvectors and classify the origin as stable/unstable node or saddle point. Calculate the solution with initial values $\mathbf{x}(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

Solution: We calculate the eigenvalues and eigenvectors of the matrix:

$$0 = \det(A - \lambda I) = \det \begin{pmatrix} 1 - \lambda & 3 \\ 1 & -1 - \lambda \end{pmatrix} = -(1 - \lambda)(1 + \lambda) - 3 = \lambda^2 - 4,$$

so the eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = -2$, with eigenvectors

$$\mathbf{v}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

In the new variables given by

$$\mathbf{x}(t) = \begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix} \mathbf{z}(t)$$

we have the decoupled system

$$\dot{\mathbf{z}}(t) = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \mathbf{z}(t).$$

and since the eigenvalues are different, the general solution is

$$\mathbf{z}(t) = Ae^{2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + Be^{-2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

and in the original variables

$$\mathbf{x}(t) = C_1 e^{2t} \begin{pmatrix} 3 \\ 1 \end{pmatrix} + C_2 e^{-2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 3C_1 e^{2t} + C_2 e^{-2t} \\ C_1 e^{2t} - C_2 e^{-2t} \end{pmatrix}.$$

Since we have two real distinct eigenvalues with opposite signs, the origin is a saddle point.

With the initial values $x_1(0) = 1$ and $x_2(0) = -1$, we have

$$3C_1 + C_2 = 1, \qquad C_1 - C_2 = -1,$$

which gives $C_1 = 0$ and $C_2 = 1$, so the particular solution is

$$\mathbf{x}(t) = \begin{pmatrix} e^{-2t} \\ -e^{-2t} \end{pmatrix}.$$

P3 (3 points) Calculate a power series solution for the following IVP, at $x_0 = 0$:

$$(1 - x)y'(x) - y(x) = 0, \qquad y(0) = 1.$$

Identify the power series, if possible, in terms of elementary functions.

Solution: Since the coefficients of the ODE can be written as power series at $x_0 = 0$, we propose a solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n,$$

and then the ODE reads

$$(1 - x) \sum_{n=0}^{\infty} (n + 1) a_{n+1} x^n - \sum_{n=0}^{\infty} a_n x^n = 0,$$

so

$$\sum_{n=0}^{\infty} (n + 1) a_{n+1} x^n - \sum_{n=0}^{\infty} (n + 1) a_{n+1} x^{n+1} - \sum_{n=0}^{\infty} a_n x^n = 0.$$

Since the first and last sums start with x^0 and the second one with x, we separate the $n = 0$ terms (and we shift the middel sum):

$$a_1 + \sum_{n=1}^{\infty} (n + 1) a_{n+1} x^n - \sum_{n=1}^{\infty} n a_n x^{n+1} - a_0 - \sum_{n=1}^{\infty} a_n x^n = 0.$$

Then we have $a_1 = a_0 = 1$, and then $a_{n+1} = a_n$ for $n \geq 1$ so all the coefficients are equal to 1, and therefore

$$y(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1 - x}, \qquad |x| < 1,$$

the geometric series.
