# EDOs-first-midterm-solved-2425.pdf

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Ampliación de Matemáticas

2º Grado en Ingeniería Aeroespacial

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Accede al documento original

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# EDOs: first midterm

General comments: Read carefully to answer all the questions and remember to frame the final answers in a nice box. The easier you make it for me, the happier I will be marking. Check the solution whenever possible (in a draft paper, no need to include it if it is not explicitly asked). Whenever you are asked to explain something in your own words, a few sentences are enough and drawings (if pertinent) are always welcome. Good luck!

1. Consider the differential equation $P(x, y(x)) + Q(x, y(x))\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$

(3) (a) Indicate if it is an ODE or PDE, its order, and if its linear or nonlinear.

(7) (b) When do we say that it is exact? Why is it useful to have exact differential equations?

(5) (c) What would you do if it was not exact?

(6) (d) Find the solution to $\ln(y(x))\sin(x) - x + \left(y - \dfrac{\cos(x)}{y(x)}\right)\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0.$

(4) (e) You should have found an implicit solution. Explain what is that, why is it worse than an explicit solution but still better than the original differential equation.

2. Consider the logistic equation $\dfrac{\mathrm{d}p}{\mathrm{d}t} = k_1 p(k_2 - p)$ for some constant $k_1, k_2 > 0$.

(5) (a) Draw the 1-dimensional phase diagram.

(5) (b) Explain what is the meaning of the constants $k_1$ and $k_2$.

(15) (c) Find the solution to the logistic equation.

3. Consider the system $\dfrac{\mathrm{d}\vec{x}}{\mathrm{d}t} = A\vec{x}$ for a real $2 \times 2$ matrix.

(10) (a) Explain how would you find the eigenvalues and eigenvectors of $A$. Why are they important?

(18) (b) Find the solution and draw the phase diagram in the following cases (assume $\dot{x} < 0$ over $\{(0, y) : y > 0\}$ whenever additional information is necessary):
i) $A$ has eigenvalues $\lambda_1 = 3$, $\lambda_2 = 1$ with eigenvectors $v_1 = \binom{1}{1}$ and $v_2 = \binom{3}{1}$.
ii) $A$ has eigenvalues $\lambda_1 = 1+3i$, $\lambda_2 = 1-3i$ with eigenvectors $v_1 = \binom{1+i}{1-i}$ and $v_2 = \binom{1-i}{1+i}$.

4. Consider the differential equation $t^2y'' + 5ty' + 3y = 0$. Notice that it is linear but not of constant coefficient. We can, however, still solve it.

(12) (a) For that, instead of using the guess $y(t) = e^{kt}$ use $y(t) = t^k$ to find the general solution.

(5) (b) Find the particular solution satisfying $y(1) = 0$ and $y'(1) = 1$.

(10) (c) Explain for which differential equations $a(t)y''' + b(t)y'' + c(t)y' + d(t)y = 0$ this method of guessing $y(t) = t^k$ will work (for instance, the guess $y(t) = e^{kt}$ works whenever $a(t), b(t), c(t)$, and $d(t)$ are constant).

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# ODE's FIRST MIDTERM

(1) $P(x, y(x)) + Q(x, y(x)) \frac{dy}{dx} = 0$

a) It is an ODE of first order (because only a first order derivative appears). Generally, they are non-linear. ~~However, it depends on $Q(x, y(x))$, if it can be written in the form $a\frac{dy}{dx} + by + c = 0$, then it will be linear~~

It would be linear if P is linear on y and Q does not depend on y

b) It will be exact if $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$. It is useful because if they are exact, the came from a potential $V(x,y)$ so that $\nabla \vec{V} = (P, Q)$ (It is not that the potential is constant for all x and y but that the solution lies on a level set) → $V(x, y(x)) = \text{constant}$

c) If it is not exact, an integrating factor that makes them exact can be found so that $\frac{\partial(\mu P)}{\partial y} = \frac{\partial(\mu Q)}{\partial x}$

d) $\underbrace{\ln y \cdot \sin x - x}_{P} + \underbrace{\left(y - \frac{\cos x}{y}\right)}_{Q} \frac{dy}{dx} = 0$

$\frac{\partial P}{\partial y} = \frac{\sin x}{y}$

$\frac{\partial Q}{\partial x} = +\frac{\sin x}{y}$

⟩ they are exact!

There exists some f such that $\nabla f = (P, Q)$

$P = \frac{\partial f}{\partial x} = \ln y \cdot \sin x - x$

$Q = \frac{\partial f}{\partial y} = y - \frac{\cos x}{y} \rightarrow \int \left(y - \frac{\cos x}{y}\right) dy =$

$= \frac{y^2}{2} - \cos x \cdot \ln y + c(x)$

$f(x,y) = \int (\ln y \cdot \sin x - x)\, dx = -\cos x \cdot \ln y - \frac{x^2}{2} + c(y)$

$\frac{\partial f}{\partial y} = -\frac{\cos x}{y} + c'(y) = y - \frac{\cos x}{y} \rightarrow c'(y) = y \rightarrow c(y) = \frac{y^2}{2} + d$

The solution is some y that satisfies

$f(x,y) = -\ln y \cdot \cos x - \frac{x^2}{2} + \frac{y^2}{2} = \text{constant}$

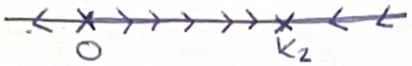
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e) An implicit solution is an algebraic equation (that requires to be solved) that provides some condition that the solution must fulfill. It is worse than a explicit solution because it is not giving you the direct result but is better than the original differential equation because no derivatives appear and it is easier to solve

(2) logistic equation $\frac{dP}{dt} = k_1 P(K_2 - P)$

a) 1-D phase diagram




$\frac{dP}{dt}$ vanishes when the population is 0 or when the max. sustainable population has been reached

b) $k_1$ = net-growth rate, that is: birthrate - deathrate

$K_2$ = maximum sustainable population (due to the availability of natural resources)

c) $\frac{dP}{dt} = k_1 P(K_2 - P) \rightarrow \frac{dP}{P(K_2-P)} = k_1 dt$

$$\frac{A}{P} + \frac{B}{K_2 - P} = \frac{A(K_2-P) + BP}{P(K_2-P)} = \frac{P(B-A) + AK_2}{P(K_2-P)} = \frac{1}{P(K_2-P)} \begin{cases} A = 1/K_2 \\ B = A \end{cases}$$

$$\frac{1}{P(K_2-P)} = \frac{1}{K_2 P} + \frac{1}{K_2(K_2-P)}$$

$$dP\left(\frac{1}{P} + \frac{1}{K_2-P}\right) = K_1 K_2 dt \rightarrow \int_{P_0}^{P(t)} \left(\frac{1}{P} + \frac{1}{K_2-P}\right) dP = \int_{t_0}^{t_F} K_1 K_2 dt$$

$$\left[\ln P - \ln(K_2 - P)\right]_{P_0}^{P(t)} = K_1 K_2 (t_F - t_0)$$

$$\left[\ln\left(\frac{P}{K_2-P}\right)\right]_{P_0}^{P(t)} = K_1 K_2 (t_F - t_0) \rightarrow \ln\left(\frac{P(t)(K_2-P_0)}{P_0(K_2-P(t))}\right) = K_1 K_2 (t_F - t_0)$$

$$\frac{P(t)}{K_2 - P(t)} = \frac{P_0}{K_2 - P_0} e^{K_1 K_2 (t_F - t_0)} \rightarrow \boxed{P(t) = \frac{K_2 \cdot P_0 \cdot e^{K_1 K_2 (t_F - t_0)}}{K_2 - P_0 + P_0 e^{K_1 K_2 (t_F - t_0)}}}$$

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③ $\frac{d\bar{x}}{dt} = A\vec{x}$

a) To find the eigenvalues we need to solve the determinant $|A - \lambda I| = 0$

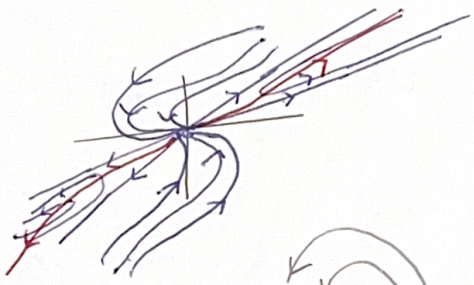
The eigenvectors can be found by solving $(A - \lambda I)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

where $v_1$ and $v_2$ are the coordinates of the eigenvector.

They are important because allow us to graphycally represent the behavior of solutions of ODE's. They are also important because we're able to decouple two ODE and give their vectorial solution in terms of the eigenvalues and eigenvectors as $x(t) = Ae^{\lambda_1 t} \cdot \bar{v}_1 + Be^{\lambda_2 t} \cdot \vec{v}_2$

They are important because they provide the invariant directions of $\frac{d\bar{x}}{dt} = A\bar{x}$ and their stability

b)



$\lambda_1 = 3 \quad ; \quad \bar{v}_1 = (1, 1)$

$\lambda_2 = 1 \quad ; \quad \bar{v}_2 = (3, 1)$

$X(t) = Ae^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + Be^{t}\begin{pmatrix} 3 \\ 1 \end{pmatrix}$



$\lambda_1 = 1 + 3i \quad ; \quad \bar{v}_1 = \begin{pmatrix} 1+i \\ 1-i \end{pmatrix}$

$\lambda_2 = 1 - 3i \quad ; \quad \bar{v}_2 = \begin{pmatrix} 1-i \\ 1+i \end{pmatrix}$

$r = 1 > 0$ unstable spiral if $\dot{x} < 0$ in $(0, y)$ counterclockwise direction

$$X(t) = e^{t}\left[(\alpha \cos 3t + \beta \sin 3t)\begin{pmatrix} 1 \\ 1 \end{pmatrix} + (-\alpha \sin 3t + \beta \cos 3t)\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right]$$

④ $t^2 y'' + 5ty' + 3y = 0$

a) Guess: $y(t) = t^k \quad ; \quad y' = kt^{k-1} \quad ; \quad y'' = k(k-1)t^{k-2}$

$$t^2 k(k-1)t^{k-2} + 5tkt^{k-1} + 3t^k = 0$$

$$k(k-1)t^k + 5kt^k + 3t^k = 0 \rightarrow k(k-1) + 5k + 3 = 0$$

$$k^2 + 4k + 3 = 0 \rightarrow k = \frac{-4 \pm \sqrt{16-12}}{2} = \frac{-4 \pm 2}{2} \begin{cases} k_1 = -1 \\ k_2 = -3 \end{cases}$$

$$y(t) = At^{-1} + Bt^{-3} = \frac{A}{t} + \frac{B}{t^3} \rightarrow y'(t) = -At^{-2} - 3Bt^{-4}$$

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b) $y(1) = 0$ we want to find A and B

$y'(1) = 1$

$$0 = A + B$$
$$+ \quad 1 = -A - 3B$$
$$1 = -2B \rightarrow B = -\frac{1}{2} \; ; \; A = \frac{1}{2}$$

$$y(t) = \frac{1}{2t} - \frac{1}{2t^3}$$

c) $a(t)y''' + b(t)y'' + c(t)y' + d(t)y = 0$

when does the guess $y = t^k$ work?

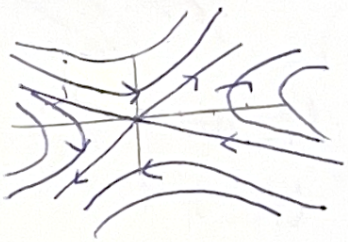
whenever the coefficients are $t$ to the power of the order of the derivative.

That is, $a(t) = \alpha t^3$ ; $b(t) = \beta t^2$ ; $c(t) = \gamma t$ and $d(t) = t^0 = 1$

(3.2)

i) $\lambda_1 = 3$ ; $\bar{v}_1 = (1,1)$

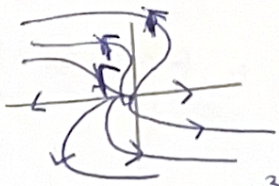
$\lambda_2 = -1$ ; $\bar{v}_2 = (3,-1)$



$$\vec{x}(t) = Ae^{3t}\begin{pmatrix}1\\1\end{pmatrix} + Be^{-t}\begin{pmatrix}3\\-1\end{pmatrix}$$

ii) $\lambda_1 = 3$

$\bar{v}_1 = \begin{pmatrix}1\\0\end{pmatrix}$



$\bar{v}_2 = (0,1)$

$$\vec{x}(t) = (Ate^{3t} + Be^{3t})\begin{pmatrix}1\\0\end{pmatrix} + Ae^{3t}\begin{pmatrix}0\\1\end{pmatrix}$$

(4.2) $t^2y'' - 3ty' + 3y = 0$

$y(t) = t^k$ ; $y' = kt^{k-1}$ ; $y'' = k(k-1)t^{k-2}$

$t^2k(k-1)t^{k-2} - 3tkt^{k-1} + 3t^k = 0 \rightarrow k(k-1) - 3k + 3 = 0$

$$k^2 - 4k + 3 = 0 \rightarrow k = \frac{4 \pm \sqrt{16-12}}{2} = \frac{4 \pm 2}{2} \begin{matrix} \nearrow 3 \\ \searrow 1 \end{matrix}$$

$y(t) = At^3 + Bt$ ; $y' = 3At^2 + B$

$0 = A + B$ ; $1 = 3A + B \rightarrow 1 = -3B + B \rightarrow B = -\frac{1}{2}$ ; $A = \frac{1}{2}$

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