EDOs: first midterm

General comments: Read carefully to answer all the questions and remember to frame the final answers in a nice box. The easier you make it for me, the happier I will be marking. Check the solution whenever possible (in a draft paper, no need to include it if it is not explicitly asked). Whenever you are asked to explain something in your own words, a few sentences are enough and drawings (if pertinent) are always welcome. Good luck!

1. Consider the differential equation $P(x, y(x)) + Q(x, y(x))\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$

(3) (a) Indicate if it is an ODE or PDE, its order, and if its linear or nonlinear.

(7) (b) When do we say that it is exact? Why is it useful to have exact differential equations?

(5) (c) What would you do if it was not exact?

(6) (d) Find the solution to $\ln(y(x))\sin(x) - x + \left(y - \dfrac{\cos(x)}{y(x)}\right)\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0.$

(4) (e) You should have found an implicit solution. Explain what is that, why is it worse than an explicit solution but still better than the original differential equation.

2. Consider the logistic equation $\dfrac{\mathrm{d}p}{\mathrm{d}t} = k_1 p(k_2 - p)$ for some constant $k_1, k_2 > 0$.

(5) (a) Draw the 1-dimensional phase diagram.

(5) (b) Explain what is the meaning of the constants k_1 and k_2.

(15) (c) Find the solution to the logistic equation.

3. Consider the system $\dfrac{\mathrm{d}\vec{x}}{\mathrm{d}t} = A\vec{x}$ for a real 2×2 matrix.

(10) (a) Explain how would you find the eigenvalues and eigenvectors of A. Why are they important?

(18) (b) Find the solution and draw the phase diagram in the following cases (assume $\dot{x} < 0$ over $\{(0, y) : y > 0\}$ whenever additional information is necessary):
i) A has eigenvalues $\lambda_1 = 3$, $\lambda_2 = 1$ with eigenvectors $v_1 = \binom{1}{1}$ and $v_2 = \binom{3}{1}$.
ii) A has eigenvalues $\lambda_1 = 1+3i$, $\lambda_2 = 1-3i$ with eigenvectors $v_1 = \binom{1+i}{1-i}$ and $v_2 = \binom{1-i}{1+i}$.

4. Consider the differential equation $t^2y'' + 5ty' + 3y = 0$. Notice that it is linear but not of constant coefficient. We can, however, still solve it.

(12) (a) For that, instead of using the guess $y(t) = e^{kt}$ use $y(t) = t^k$ to find the general solution.

(5) (b) Find the particular solution satisfying $y(1) = 0$ and $y'(1) = 1$.

(10) (c) Explain for which differential equations $a(t)y''' + b(t)y'' + c(t)y' + d(t)y = 0$ this method of guessing $y(t) = t^k$ will work (for instance, the guess $y(t) = e^{kt}$ works whenever $a(t), b(t), c(t)$, and $d(t)$ are constant).

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(10) (a) Explain how would you find the eigenvalues and eigenvectors of A. Why are they important?

(18) (b) Find the solution and draw the phase diagram in the following cases (assume $\dot{x} < 0$ over $\{(0, y) : y > 0\}$ whenever additional information is necessary):
i) A has eigenvalues $\lambda_1 = 3$, $\lambda_2 = -1$ with eigenvectors $v_1 = \binom{1}{1}$ and $v_2 = \binom{3}{-1}$.
ii) A has eigenvalues $\lambda_1 = 3$ with eigenvector $v_1 = \binom{1}{0}$

4. Consider the differential equation $t^2y'' - 3ty' + 3y = 0$. Notice that it is linear but not of constant coefficient. We can, however, still solve it.

(12) (a) For that, instead of using the guess $y(t) = e^{kt}$ use $y(t) = t^k$ to find the general solution.

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