ADVANCED MATHEMATICS

Bachelor Degree in Aerospace Engineering, Universidad Carlos III de Madrid

Group 45, course 2021-2022

October 29th, 2021

Total duration: 1.5 hours

Name:

P1 (3 points) A certain population $P(t)$ (in millions) evolves in time according to the following logistic model:

$$P'(t) = P(t) - \tfrac{1}{2}P(t)^2.$$

1. Calculate the stationary solutions of this autonomous ODE.
2. Determine the population when $t = 2$ if the initial population at time $t_0 = 0$ is $P_1(0) = 1$. Same question if the initial value is $P_2(0) = 3$. Plot these two solutions, together with the stationary ones.
3. Calculate the limit behavior of the two solutions $\lim_{t\to\infty} P_1(t)$ and $\lim_{t\to\infty} P_2(t)$.

Solution:

1. The stationary solutions correspond to the zeros of $f(P)$, where $f(P) = P - \frac{1}{2}P^2 = P\left(1 - \frac{1}{2}P\right)$, so they are $P(t) = 0$ and $P(t) = 2$. These are stationary because for these solutions we have $P'(t) = 0$ and they are constant in time.

2. This is a separable ODE, so we write

$$\frac{1}{P(t)(1 - \frac{1}{2}P(t))}P'(t) = 1.$$

Partial fractions on the left gives

$$\left(\frac{1}{P(t)} + \frac{\frac{1}{2}}{1 - \frac{1}{2}P(t)}\right)P'(t) = 1.$$

Now we integrate and we obtain

$$\ln|P(t)| - \ln|1 - \tfrac{1}{2}P(t)| = t + C \Rightarrow \frac{P(t)}{1 - \frac{1}{2}P(t)} = \tilde{C}e^t.$$

If $P_1(0) = 1$, then we can determine the constant $\tilde{C}$:

$$\frac{1}{1 - \frac{1}{2}} = 2 = \tilde{C},$$

so the solution is

$$P_1(t) = \frac{2e^t}{1 + e^t} = \frac{2}{1 + e^{-t}}.$$

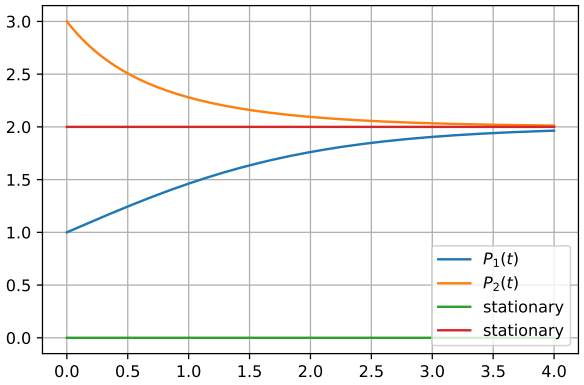


Figure 1: Solution and stationary ones.

It follows that when $t = 2$ we have $P_1(2) \approx 1.7616$.

If $P_2(0) = 3$, then

$$P_2(t) = \frac{6}{3 - e^{-t}},$$

so $P_2(2) \approx 2.0949$.

3. From the previous formulas we see that

$$\lim_{t\to\infty} P_1(t) = \lim_{t\to\infty} P_2(t) = 2.$$

Both solutions tend to the stationary one $P(t) = 2$ as $t \to \infty$. This can be shown with the existence and uniqueness theorem, together with the sign of $P'(t)$ for different initial values.

P2 (2+1 points)

1. Compute the solution of the following initial value problems:

$$y''(x) - 6y'(x) + 9y(x) = e^{3x}, \qquad y(0) = 1, \quad y'(0) = 0.$$

2. Determine for which value(s) of $a \in \mathbb{R}$ the following ODE is exact, and compute the solution in that case:

$$2xy - y^2 + (1 + x^2 + axy)\frac{dy}{dx} = 0, \qquad y(1) = 1.$$

Solution:

1. The general solution of this ODE is $y(x) = y_h(x) + y_p(x)$, where $y_h(x)$ is the general solution of the homogeneous ODE and $y_p(x)$ is a particular one of the inhomogeneous one.

In this case, $y_h''(x)-6y_h'(x)+9y_h(x)=0$, so if $y_h(x)=e^{\lambda x}$, we obtain $e^{\lambda x}(\lambda^2-6\lambda+9)=0$. Since $e^{\lambda x}\neq 0$, we solve $\lambda^2-6\lambda+9=0$, which gives a double solution $\lambda=3$. Therefore, we have

$$y_h(x)=c_1e^{3x}+c_2xe^{3x},$$

for some constants c_1 and c_2.

To calculate $y_p(x)$, we can apply undetermined coefficients. Note that e^{3x} and xe^{3x} already solve the homogeneous ODE, so we need to try a different function. For example,

$$y_p(x)=Ax^2e^{3x},$$

then

$$\begin{aligned} y_p'(x)&=(2Ax+3Ax^2)e^{3x},\\ y_p''(x)&=(2A+12Ax+9Ax^2)e^{3x}, \end{aligned}$$

and then

$$\begin{aligned} y_p''(x)-6y_p'(x)+9y_p(x)&=\left(2A+12Ax+9Ax^2-12Ax-18Ax^2+9Ax^2\right)e^{3x}\\ &=2Ae^{3x}, \end{aligned}$$

and we can determine $A=1/2$. Therefore

$$y_p(x)=\frac{1}{2}x^2e^{3x},$$

and the general solution of the ODE is

$$y(x)=c_1e^{3x}+c_2xe^{3x}+\frac{1}{2}x^2e^{3x}$$

If $y(0)=1$, we have $1=c_1$, and if $y'(0)=0$ we get $0=3c_1+c_2$, so $c_2=-3$. Therefore the solution is

$$y(x)=e^{3x}-3xe^{3x}+\frac{1}{2}x^2e^{3x}.$$

2. In this ODE we have $f(x,y)=2xy-y^2$ and $g(x,y)=1+x^2+axy$, so $M_y=2x-2y$ and $g_x=2x+ay$, so it is exact if $a=-2$. In this case, the general solution is $F(x,y)=C$, where

$$\begin{aligned} \frac{\partial F}{\partial x}&=f(x,y)\Rightarrow F(x,y)=x^2y-xy^2+h(y)\\ \frac{\partial F}{\partial y}&=g(x,y)\Rightarrow F(x,y)=y+x^2y-xy^2+m(x) \end{aligned}.$$

Comparing these equations, we obtain $F(x,y)=x^2y-xy^2+y$, so the general solution is

$$x^2y-xy^2+y=C.$$

If $y(1)=1$, we obtain $C=1$, so the particular solution is $x^2y-xy^2+y=1$.

P3 (4 points) Consider the following linear system of ODEs:

$$\frac{d}{dt}\mathbf{x}(t)=\begin{pmatrix}1&2\\1&0\end{pmatrix}\mathbf{x}(t),\qquad \mathbf{x}(t)=\begin{pmatrix}x_1(t)\\x_2(t)\end{pmatrix}$$

1. Write the general solution of the system.
2. Classify the type of stationary point at the origin and draw the phase portrait.
3. Compute the solution with initial values

$$\mathbf{x}(0) = \begin{pmatrix} 0 \\ 1/2 \end{pmatrix}.$$

Solution:

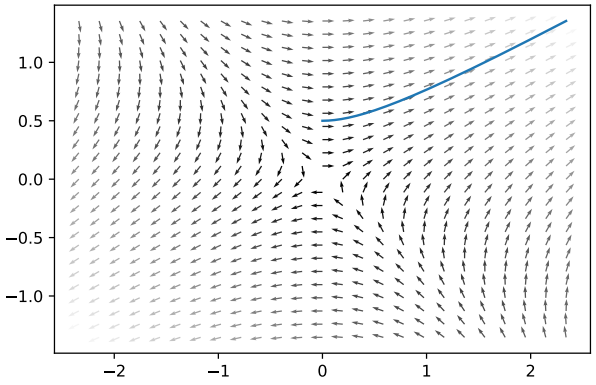


Figure 2: Phase portait and trajectory with given initial value.

1. The matrix

$$A = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$$

has eigenvalues given by $0 = (2-\lambda)(-2-\lambda)+5 = \lambda^2+1$, so $\lambda = \pm i$.
The eigenvectors are

$$\eta_+ = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \qquad \eta_- = \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

so the general solution is

$$\mathbf{x}(t) = c_1 e^{2t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

2. Since A has two real eigenvalues with different signs, the origin is a saddle point.
3. If we impose the initial values at $t = 0$, we have

$$\begin{pmatrix} 0 \\ 1/2 \end{pmatrix} = \begin{pmatrix} 2c_1 + c_2 \\ c_1 - c_2 \end{pmatrix},$$

so $c_1 = 1/6$ and $c_2 = -1/3$, the particular solution is

$$\mathbf{x}(t) = e^{2t} \begin{pmatrix} 1/3 \\ 1/6 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} -1/3 \\ 1/3 \end{pmatrix} = \frac{1}{6} \begin{pmatrix} 2e^{2t} - 2e^{-t} \\ e^{2t} + 2e^{-t} \end{pmatrix}.$$