ADVANCED MATHEMATICS

Bachelor Degree in Aerospace Engineering, Universidad Carlos III de Madrid

Group 47, course 2020-2021

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Total duration: 1 hour

P1 (2 points) We consider the following initial value problem:

$$y' = y + \frac{x}{y}, \qquad y(0) = 1.$$

Make the change of variable $t(x) = y(x)^2$ and find the solution.

Solution: If $t(x) = y(x)^2$, then $t' = 2yy'$, and therefore

$$y' = \frac{t'}{2y} = \frac{t'}{2t^{1/2}},$$

so the ODE becomes

$$\frac{t'}{2t^{1/2}} = t^{1/2} + \frac{x}{t^{1/2}} \Rightarrow t' = 2t + 2x,$$

which is a linear ODE. With the integrating factor $\mu(x) = e^{-2x}$, we obtain

$$(te^{-2x})' = 2xe^{-2x} \Rightarrow te^{-2x} = \int 2xe^{-2x}dx.$$

We apply integration by parts with $u = x$ and $dv = 2e^{-2x}$, so

$$\int 2xe^{-2x}dx = -xe^{-2x} + \int e^{-2x}dx = -\left(x + \frac{1}{2}\right)e^{-2x} + C,$$

and the general solution is

$$t = -x - \frac{1}{2} + Ce^{2x} \Rightarrow y^2 = -x - \frac{1}{2} + Ce^{2x}.$$

If $y(0) = 1$, then we obtain $1 = -\frac{1}{2} + C$, so $C = \frac{3}{2}$ and the particular solution is

$$y = \sqrt{-x - \frac{1}{2} + \frac{3}{2}e^{2x}}.$$

P2 (8 points) Given the following system of differential equations

$$\dot{\mathbf{x}}(t) = M\mathbf{x}(t), \qquad M = \begin{pmatrix} -3 & 2 \\ 1 & -2 \end{pmatrix}, \qquad \mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix},$$

- (6 points) Calculate the general solution, classify the origin as stable/unstable node or saddle point and draw the phase diagram.

- (2 points) Transform the inhomogeneous system

$$\dot{\mathbf{v}}(t) = M\mathbf{v}(t) + \begin{pmatrix} 0 \\ e^{-t} \end{pmatrix}, \qquad \mathbf{v}(t) = \begin{pmatrix} v(t) \\ w(t) \end{pmatrix}$$

into a second order ODE for $w(t)$, solve it and obtain the solutions of the inhomogeneous system $v(t)$ and $w(t)$.

Solution:

- We calculate the eigenvalues and eigenvectors of the matrix:

$$0 = \det(A - \lambda I) = \det \begin{pmatrix} -3-\lambda & 2 \\ 1 & -2-\lambda \end{pmatrix} = (3+\lambda)(2+\lambda) - 2 = \lambda^2 + 5\lambda + 4,$$

so the eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = -4$, with eigenvectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix},$$

In the new variables given by

$$\mathbf{x}(t) = \begin{pmatrix} -3 & 2 \\ 1 & -2 \end{pmatrix} \widetilde{\mathbf{x}}(t)$$

we have the decoupled system

$$\dot{\widetilde{\mathbf{x}}}(t) = \begin{pmatrix} -1 & 0 \\ 0 & -4 \end{pmatrix} \widetilde{\mathbf{x}}(t).$$

Therefore, the general solution is

$$\widetilde{\mathbf{x}}(t) = Ae^{-t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + Be^{-4t} \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

and in the original variables

$$\mathbf{x}(t) = Ae^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + Be^{-4t} \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

Since we have two real distinct eigenvalues with negative signs, the origin is a stable node, and the phase portrait looks as in the figure.

- To transform the inhomogeneous system into a second order ODE, we write

$$v'(t) = -3v(t) + 2w(t), \qquad w'(t) = v(t) - 2w(t) + e^t,$$

so $v(t) = w'(t) + 2w(t) - e^{-t}$ in the second equation, and replacing in the first one we have

$$w''(t) + 2w'(t) + e^{-t} = -3w'(t) - 6w(t) + 3e^{-t} + 2w(t) \Rightarrow w''(t) + 5w'(t) + 4w(t) = 2e^{-t}.$$

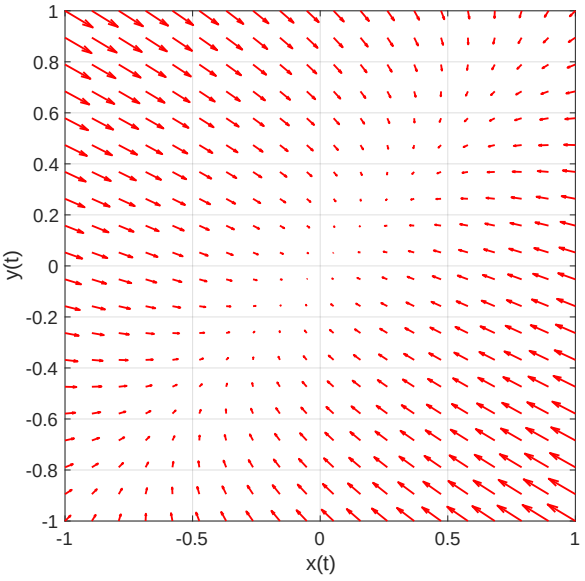


Figure 1: Phase portrait for Exercise 2.

We know that the general solution of the homogenous ODE is $w_h(t) = Ae^{-t} + Be^{-4t}$, so for the particular solution we try

$$w_p(t) = Cte^{-t} \Rightarrow w_p''(t) + 5w_p'(t) + 4w_p(t) = 3Ce^{-t},$$

so we need $C = \frac{2}{3}$. The general solution is then

$$w(t) = Ae^{-t} + Be^{-4t} + \frac{2}{3}te^{-t},$$

and therefore

$$v(t) = w'(t) + 2w(t) - e^{-t} = \left(A - \frac{1}{3}\right)e^{-t} - 2Be^{-4t} + \frac{2}{3}te^{-t}.$$
