ADVANCED MATHEMATICS

Bachelor Degree in Aerospace Engineering, Universidad Carlos III de Madrid

Group 46, course 2020-2021

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Total duration: 1 hour

PROBLEM 1. (5 points)

Consider the differential equation

$$xy'' + y' = 4x,$$

where $y \equiv y(x)$ and $y'(x) = \frac{dy(x)}{dx}$.

1.1 Find the general solution for such a differential equations.

HINT: You might use a reduction of order.

1.2 Using the existence and uniqueness Theorem justify the domain for which one can find a solution.

Solution:

1.1 Performing the change of variable $y' = p$, such that $y'' = p'$ we arrive at the equation

$$xp' + p = 4x.$$

Hence, since the left hand side is $xp' + p = \frac{d}{dx}(xp)$, integrating we find that

$$\frac{d}{dx}(xp) = 4x \Rightarrow xp(x) = \int 4x\, dx = 2x^2 + C_1.$$

Then,

$$p(x) = 2x + \frac{C_1}{x}, \quad \text{if} \quad x \neq 0,$$

with $C_1 \in \mathbb{R}$. Also, since $y' = p$ we finally obtain that

$$y(x) = \int \left(2x + \frac{C_1}{x}\right) dx = x^2 + C_1 \log|x| + C_2, \quad \text{with} \quad C_1, C_2 \in \mathbb{R}.$$

1.2 To use the existence and uniqueness Theorem we might write the ODE as

$$y'' = 4 - \frac{1}{x}y' = f(x, y').$$

Thus, we just have to justify the continuity of f with respect to all the arguments x and y', which will be true if and only if $x \neq 0$. Something consistent with the explicit solution obtained above. And also, the continuity of the partial derivative of f with respect to y', i.e. $\frac{\partial f(y')}{\partial y'}$. Which it will be achieved again if $x \neq 0$.

Therefore, thanks to the existence an uniqueness Theorem we can conclude that there exists a unique solution in the interval (a, b) if $0 \notin (a, b)$.

PROBLEM 2. (5 points)

Consider the system of equations

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -2 & -4 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1+4t \\ \frac{3}{2}t^2 \end{pmatrix},$$

with $\frac{dx}{dt} = x'$ and $\frac{dy}{dt} = y'$.

2.1 Write down the general solution for the homogeneous associated system, leaving it in terms of two arbitrary real constants.

2.2 Sketch the phase portrait. What type of stationary point is the origin? Find the coordinate transformation that puts the homogenous system into canonical form.

2.3 Find the solution for the non-homogeneous system.

You might use the following integrals, obtained after integration by parts:

$$\int te^{kt}\,dt = \frac{e^{kt}}{k^2}(kt-1) + C,$$

and

$$\int t^2 e^{kt}\,dt = e^{kt}\left(\frac{t^2}{k} - \frac{2t}{k^2} + \frac{2}{k^3}\right) + C,$$

with C a real constant.

Solution:

2.1 First we solve the homogeneous associated problem

$$\begin{cases} x' + 2x + 4y = 0 \\ y' + x - y = 0 \end{cases}$$

Differentiating the first equation with respect to t it follows that

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 4\frac{dy}{dt} = 0.$$

Also, from the system (second and first equation respectively) we have that

$$\frac{dy}{dt} = y - x \quad \text{and} \quad 4y = -\frac{dx}{dt} - 2x,$$

and, then, substituting them into the derivative of the first equation we find that

$$\frac{d^2x}{dt^2} + \frac{dx}{dt} - 6x = 0.$$

The general solution for this second order ODE with constant coefficients is

$$x(t) = C_1e^{2t} + C_2e^{-3t}, \quad \text{with} \quad C_1, C_2 \in \mathbb{R}.$$

Moreover, since

$$4y = -\frac{dx}{dt} - 2x,$$

we have that

$$y(t) = -C_1e^{2t} + \frac{C_2}{4}e^{-3t}.$$

2.2 Computing the determinant $\det(A-\lambda \mathrm{I})$ for the matrix

$$\begin{pmatrix} -2 & -4 \\ -1 & 1 \end{pmatrix},$$

and finding the roots for the characteristic polynomial $\det(A-\lambda \mathrm{I})=0$ we obtain that there are two real eigenvalues $\lambda_1=2$ and $\lambda_2=-3$. Thus, the eigenvectors come from the basis of the null spaces of $(A-2\mathrm{I})$ and $(A+3\mathrm{I})$ such that, if $\lambda=2$

$$\begin{pmatrix} -4 & -4 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

and the eigenvector spanning the eigenspace $\mathrm{Nul}(A-2\mathrm{I})$ is $\mathbf{v}_1=(1,-1)^T$. On the other hand, if $\lambda=-3$

$$\begin{pmatrix} 1 & -4 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

and the eigenvector spanning the eigenspace $\mathrm{Nul}(A+3\mathrm{I})$ is $\mathbf{v}_2=(4,1)^T$. Therefore, since we have two different real eigenvalues with opposite sign the singular point will be a **saddle point**.

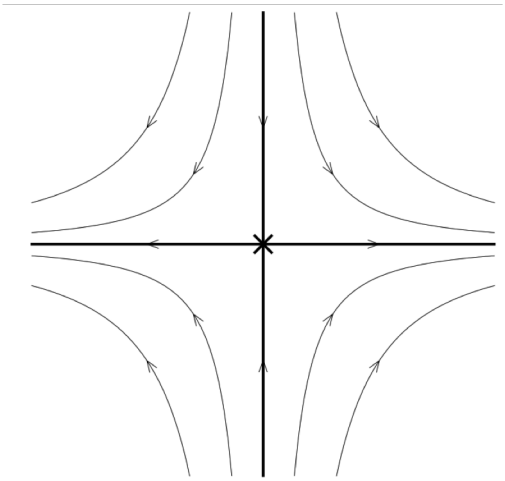


Figure 1: Saddle point.

Furthermore, due to the change of coordinates $\mathbf{x}=P\tilde{\mathbf{x}}=[\mathbf{v}_1\ \mathbf{v}_2]\tilde{\mathbf{x}}$ we find a decoupled system. Indeed

$$\frac{d\mathbf{x}}{dt}=P\frac{d\tilde{\mathbf{x}}}{dt}=AP\tilde{\mathbf{x}}=A[\mathbf{v}_1\ \mathbf{v}_2]\tilde{\mathbf{x}},$$

where

$$P=[\mathbf{v}_1\ \mathbf{v}_2]=\begin{pmatrix} 1 & 4 \\ -1 & 1 \end{pmatrix},$$

and since P is invertible, there exists $P^{-1}=[\mathbf{v}_1\ \mathbf{v}_2]^{-1}$ so that multiplying by the inverse from the left it follows that

$$P^{-1}P\frac{d\tilde{\mathbf{x}}}{dt}=\frac{d\tilde{\mathbf{x}}}{dt}=[\mathbf{v}_1\ \mathbf{v}_2]^{-1}A[\mathbf{v}_1\ \mathbf{v}_2]\tilde{\mathbf{x}}=\begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}\tilde{\mathbf{x}}.$$

In the new coordinates our original system becomes

$$\begin{cases} \frac{d\tilde{x}}{dt}=2\tilde{x}, \\ \frac{d\tilde{y}}{dt}=-3\tilde{y}, \end{cases}$$

so that

$$\tilde{x}(t)=C_1e^{2t},\quad \tilde{y}(t)=C_2e^{-3t},\quad \text{with}\quad C_1,C_2\in\mathbb{R},$$

and

$$\mathbf{x}(t) = C_1 e^{2t}\mathbf{v}_1 + C_2 e^{-3t}\mathbf{v}_2 = C_1 e^{2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + C_2 e^{-3t}\begin{pmatrix} 4 \\ 1 \end{pmatrix},$$

as obtained above in item 1.1.

2.3 To obtain the solution for the non-homogenous problem we use the method of variation of constants. To simplify the solutions of the homogenous problem we change the constant C_2 by $4C_2$. Thus,

$$\begin{cases} x(t) = C_1 e^{2t} + 4C_2 e^{-3t}, \\ y(t) = -C_1 e^{2t} + C_2 e^{-3t}. \end{cases}$$

Substituting it into the non-homogeneous system

$$\begin{cases} C_1' e^{2t} + 4C_2' e^{-3t} = 1 + 4t, \\ -C_1' e^{2t} + C_2' e^{-3t} = \frac{3}{2}t^2, \end{cases}$$

so that

$$C_1'(t) = \frac{1 + 4t - 6t^2}{5} e^{-2t},$$

$$C_2'(t) = \frac{1 + 4t + \frac{3}{2}t^2}{5} e^{3t}.$$

Consequently,

$$C_1(t) = \frac{1 + 3t^2}{5} e^{-2t} + C_1,$$

$$C_2(t) = \frac{1 + \frac{1}{2}t^2}{5} e^{3t} + C_2,$$

and the solution of the non-homogeneous will be

$$x(t) = C_1 e^{2t} + 4C_2 e^{-3t} + t + t^2,$$

and

$$y(t) = -C_1 e^{2t} + C_2 e^{-3t} - \frac{1}{2}t^2,$$

with $C_1, C_2 \in \mathbb{R}$.