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**P1** (4 points)

a) (2 points) By making the change of variable $x = e^z$ and using the chain rule, show that the second order ODE

$$x^2y''(x) + pxy'(x) + qy(x) = g(x),$$

where $p, q$ are constants, is transformed into one with constant coefficients in the variable $z$:

$$\frac{d^2y}{dz^2} + (p-1)\frac{dy}{dz} + y = g(e^z).$$

b) (2 points) Take $p = 3$, $q = -3$ and $g(x) = 4x$, make the previous transformations and compute the general solution.

**Solution:**

a) If $x = e^z$, then $z = \ln x$, and by the chain rule

$$y'(x) = \frac{dy}{dz}\frac{dz}{dx} = \frac{1}{x}\frac{dy}{dz},$$
$$y''(x) = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right) = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d^2y}{dz^2}\frac{dz}{dx} = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x^2}\frac{d^2y}{dz^2},$$

and then the ODE becomes

$$x^2\left(-\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x^2}\frac{d^2y}{dz^2}\right) + px\cdot\frac{1}{x}\frac{dy}{dz} + qy = g(e^z) \Rightarrow \frac{d^2y}{dz^2} + (p-1)\frac{dy}{dz} + qy = g(e^z),$$

as required.

b) In this case we arrive at the ODE

$$\frac{d^2y}{dz^2} + 2\frac{dy}{dz} - 3y = 4e^z.$$

We compute first the general solution of the homogeneous ODE:

$$\frac{d^2y_h}{dz^2} + 2\frac{dy_h}{dz} - 3y_h = 0.$$

Setting $y_h(z) = e^{\lambda z}$, we obtain $e^{\lambda z}(\lambda^2+2\lambda-3) = 0$, with solutions $\lambda = 1$ and $\lambda = -3$. Therefore, we have

$$y_h(z) = c_1e^z + c_2e^{-3z}.$$

To obtain a particular solution of the inhomogeneous ODE, we use variation of constants: $y_p(z) = c_1(z)e^z + c_2(z)e^{-3z}$, and then we have the following equations for this method:

$$c_1'(z)e^z + c_2'(z)e^{-3z} = 0, \qquad c_1'(z)e^z - 3c_2'(z)e^{-3z} = 0.$$

The first one gives $c_1'(z) = -c_2'(z)4^{-4z}$, and substituting into the second we find $c_2'(z) = -e^{4z}$, so $c_2(z) = -\frac{1}{4}e^{4z}$. Therefore $c_1'(z) = 1$ and $c_1(z) = z$, and then

$$y_p(z) = ze^z - \frac{1}{4}e^{-3z}.$$

The general solution is then

$$y(z) = y_h(z) + y_p(z) = c_1e^z + c_2e^{-3z} + ze^z,$$

and in the original variable

$$y(x) = c_1x + c_2x^{-3} + x\ln x.$$

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**P2** (6 points) Given the following system of differential equations

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t), \qquad A = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix}, \qquad \mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix},$$

calculate the general solution, classify the origin as stable/unstable node or saddle point and draw the phase diagram.

**Solution:** We calculate the eigenvalues and eigenvectors of the matrix:

$$0 = \det(A - \lambda I) = \det\begin{pmatrix} 1-\lambda & 3 \\ 1 & -1-\lambda \end{pmatrix} = -(1-\lambda)(1+\lambda) - 3 = \lambda^2 - 4,$$

so the eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = -2$, with eigenvectors

$$\mathbf{v}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

In the new variables given by

$$\mathbf{x}(t) = \begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix}\widetilde{\mathbf{x}}(t)$$

we have the decoupled system

$$\dot{\widetilde{\mathbf{x}}}(t) = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}\widetilde{\mathbf{x}}(t).$$

Therefore, the general solution is

$$\widetilde{\mathbf{x}}(t) = Ae^{2t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + Be^{-2t}\begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

and in the original variables

$$\mathbf{x}(t) = Ae^{2t}\begin{pmatrix} 3 \\ 1 \end{pmatrix} + Be^{-2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Since we have two real distinct eigenvalues with opposite signs, the origin is a saddle point, and the phase portrait looks as follows:

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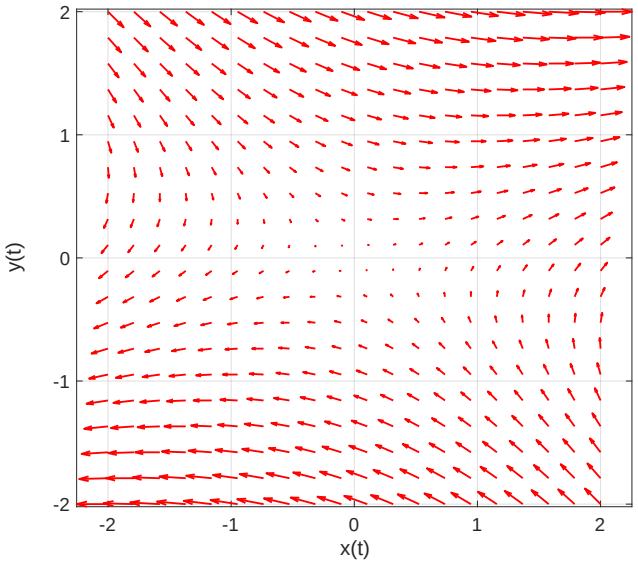



Figure 1: Phase portrait for Exercise 2.