ADVANCED MATHEMATICS

Bachelor Degree in Aerospace Engineering, Universidad Carlos III de Madrid

October 23th, 2019

Total duration: 1 hour

PROBLEM 1. (4 points)

Given $\mathbf{x}(t) = (x(t),\ y(t))^T$, consider the system of equations

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} -3 & 1 \\ -1 & -1 \end{pmatrix} \mathbf{x}.$$

1.1 Find the eigenvalues and eigenvectors of the coefficient matrix occurring in the system of equations and sketch the phase portrait.

1.2 Find the coordinate transformation that puts the system into canonical form and show that this works.

1.3 Write down the general solution.

SOLUTION:

1.1 Computing the determinant $\det(A - \lambda \mathrm{I})$ for the matrix

$$\begin{pmatrix} -3 & 1 \\ -1 & -1 \end{pmatrix},$$

and finding the roots for the characteristic polynomial $\det(A - \lambda \mathrm{I}) = 0$ we obtain that there is just one double eigenvalue $\lambda = -2$ acting as the only eigenvalue for the matrix A. Thus, the eigenvectors come from the basis of the null space of $(A + 2\mathrm{I})$ such that

$$\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

and the eigenvector spanning the eigenspace $\mathrm{Nul}(A + 2\mathrm{I})$ is $\mathbf{v} = (1, 1)^T$. We have that $\lambda = -2$ is the only eigenvalue associated with the eigenvector $\mathbf{v} = (1, 1)^T$. Now, we choose any vector $\mathbf{v}_2$ that is not in the same direction as $\mathbf{v}$. For example the perpendicular one $\mathbf{v}_2 = (1, -1)^T$. Since these two vectors are linearly independent they form a basis for $\mathbb{R}^2$, and we can write any vector $\mathbf{x}$ as a linear combination of $\mathbf{v}_2$ and $\mathbf{v}$. Then, we find the other vector in the same direction of $\mathbf{v}$

$$\mathbf{v}_1 = (A + 2\mathrm{I})\mathbf{v}_2 = \begin{pmatrix} -2 \\ -2 \end{pmatrix}.$$

1.2 Assuming the change of coordinates $\mathbf{x} = P\tilde{\mathbf{x}} = [\mathbf{v}_1\ \mathbf{v}_2]\tilde{\mathbf{x}}$ we find

$$\frac{d\mathbf{x}}{dt} = P\frac{d\tilde{\mathbf{x}}}{dt} = AP\tilde{\mathbf{x}} = A[\mathbf{v}_1\ \mathbf{v}_2]\tilde{\mathbf{x}},$$

where

$$P = [\mathbf{v}_1\ \mathbf{v}_2] = \begin{pmatrix} -2 & 1 \\ -2 & -1 \end{pmatrix},$$

and since P is invertible, there exists $P^{-1} = [\mathbf{v}_1\ \mathbf{v}_2]^{-1}$ so that multiplying by the inverse from the left it follows that

$$P^{-1}P\frac{d\tilde{\mathbf{x}}}{dt} = \frac{d\tilde{\mathbf{x}}}{dt} = [\mathbf{v}_1\ \mathbf{v}_2]^{-1}A[\mathbf{v}_1\ \mathbf{v}_2]\tilde{\mathbf{x}}.$$

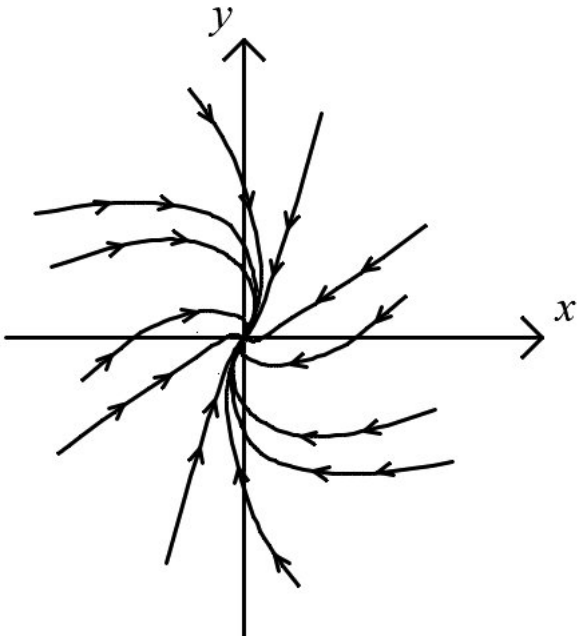


Figure 1: Stable improper node.

Moreover to write the system down we use the fact that

$$(A + 2\mathrm{I})\mathbf{v}_1 = 0 \Rightarrow A\mathbf{v}_1 = -2\mathbf{v}_1.$$

Moreover,

$$\mathbf{v}_1 = (A + 2\mathrm{I})\mathbf{v}_2 \Rightarrow A\mathbf{v}_2 = \mathbf{v}_1 - 2\mathbf{v}_2.$$

In other words,

$$A[\mathbf{v}_1\ \mathbf{v}_2] = [\lambda\mathbf{v}_1\ \mathbf{v}_1 - 2\mathbf{v}_2] = [\mathbf{v}_1\ \mathbf{v}_2]\begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix}.$$

In the new coordinates our original equation becomes

$$\frac{d\tilde{\mathbf{x}}}{dt} = \begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix}\tilde{\mathbf{x}}.$$

1.3 The general solution for the system

$$\begin{cases} \frac{d\tilde{x}}{dt} = -2\tilde{x} + \tilde{y}, \\ \frac{d\tilde{y}}{dt} = -2\tilde{y}, \end{cases}$$

so that

$$\tilde{x}(t) = (A + Bt)e^{-2t}, \quad \tilde{y}(t) = Be^{-2t}, \quad \text{with} \quad A, B \in \mathbb{R},$$

and

$$\mathbf{x}(t) = (A + Bt)e^{-2t}\mathbf{v}_1 + Be^{-2t}\mathbf{v}_2 = (A + Bt)e^{-2t}\begin{pmatrix} -2 \\ -2 \end{pmatrix} + Be^{-2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

PROBLEM 2. (6 points)

Consider the following ecological model of two interacting species whose populations are x and y:

$$\begin{aligned} \frac{dx}{dt} &= x(4 - 2x + \alpha y), \\ \frac{dy}{dt} &= y(9 - 3y). \end{aligned}$$

2.1 Describe the interaction between the species for $\alpha < 0$, $\alpha = 0$ and $\alpha > 0$.

For items 2.2 to 2.4, we only consider the specific case in which $\alpha = 1$.

2.2 Find the stationary points and determine their type.

2.3 Sketch the phase portrait.

2.4 Based on the phase portrait and your answer to 2.1, discuss the evolution of the two populations.

SOLUTION:

1.1 While y always evolves independently of x, x is influenced by y whenever $\alpha \neq 0$. The presence of y is beneficial to x for $\alpha > 1$, and detrimental for $\alpha < 1$.

1.2 We must look for (x, y) pairs that satisfy

$$\begin{aligned} x(4 - 2x + y) &= 0, \\ y(9 - 3y) &= 0. \end{aligned}$$

The second equation is only satisfied for $y = 0$ and $y = 3$. In the first equation, for $y = 0$ we obtain $x = 0$ and $x = 2$, whereas for $y = 3$ we obtain $x = 0$ and $x = 7/2$. The four stationary points are thus $(0, 0)$, $(0, 3)$, $(2, 0)$ and $(7/2, 3)$.

The Jacobian matrix that gives the linearization of the system around a given point (x, y) is

$$D\mathbf{f}(x, y) = \begin{pmatrix} 4 - 4x + y & x \\ 0 & 9 - 6y \end{pmatrix}$$

At $(0, 0)$

$$D\mathbf{f}(0, 0) = \begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix}$$

which has as eigenvalues $\lambda_1 = 4$ and $\lambda_2 = 9$. This is an unstable node.

At $(0, 3)$

$$D\mathbf{f}(0, 3) = \begin{pmatrix} 7 & 0 \\ 0 & -9 \end{pmatrix}$$

which has as eigenvalues $\lambda_1 = 7$ and $\lambda_2 = -9$. This is a saddle point.

At $(2, 0)$

$$D\mathbf{f}(2, 0) = \begin{pmatrix} -4 & 2 \\ 0 & 9 \end{pmatrix}$$

which has as eigenvalues $\lambda_1 = -4$ and $\lambda_2 = 9$. This is a saddle point too.

At $(7/2, 3)$

$$D\mathbf{f}(7/2, 3) = \begin{pmatrix} -7 & \frac{7}{2} \\ 0 & -9 \end{pmatrix}.$$

which has as eigenvalues $\lambda_1 = -7$ and $\lambda_2 = -9$. This is a stable node.

1.3 The eigenvectors associated with the origin $(0, 0)$ are $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ (corresponding to $\lambda_1 = 4$) and $\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ (corresponding to $\lambda_2 = 9$).

The eigenvectors associated with $(0,3)$ are $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ (corresponding to $\lambda_1 = 7$) and $\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ (corresponding to $\lambda_2 = -9$).

The eigenvectors associated with $(2,0)$ are $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ (corresponding to $\lambda_1 = -4$) and $\mathbf{v}_2 = \begin{pmatrix} \frac{2}{13} \\ 1 \end{pmatrix}$ (corresponding to $\lambda_2 = 9$).

The eigenvectors associated with $(7/2,3)$ are $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ (corresponding to $\lambda_1 = -7$) and $\mathbf{v}_2 = \begin{pmatrix} -\frac{7}{4} \\ 1 \end{pmatrix}$ (corresponding to $\lambda_2 = -9$).

Point	λ_1	$\mathbf{v}_1$	λ_2	$\mathbf{v}_2$	Stability type
$(0,0)$	4	$(1,0)^T$	9	$(0,1)^T$	unstable node
$(0,3)$	7	$(1,0)^T$	-9	$(0,1)^T$	saddle point
$(2,0)$	-4	$(1,0)^T$	9	$(2/13,1)^T$	saddle point
$(7/2,3)$	-7	$(1,0)^T$	-9	$(-7/4,1)^T$	stable node

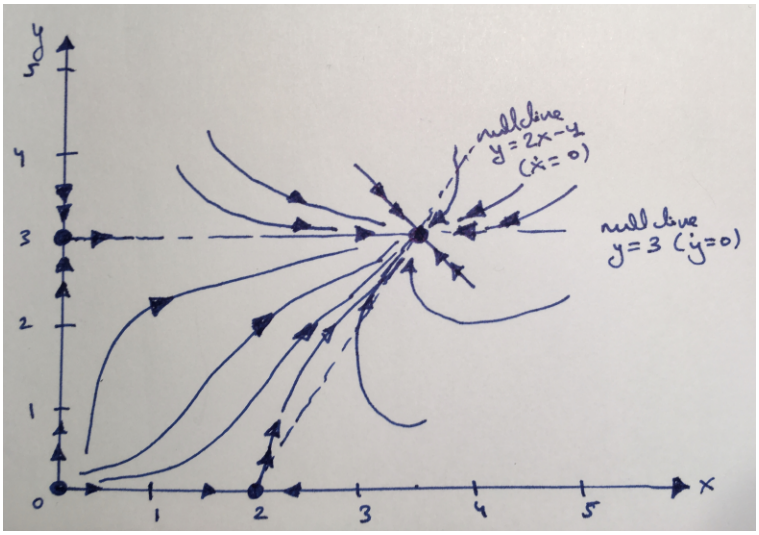


Figure 2: Phase portrait for $\alpha = 1$.

1.4 All trajectories that start from $x(0), y(0) > 0$ tend towards the stable node $(7/2,3)$, where both species coexist. The evolution of y is independent of x, as shown by the fact that for trajectories with $x(0) = 0$ we also have $y \to 3$. On the other hand, the stationary value of the population x is increased with respect to the $y(0) = 0$ case, which is in agreement with our expectations as explained in the answer to 1.1.