P.1

$$\frac{dU}{dt} = -K[U - T(t)] \qquad T(t) = T_0 + Ct \qquad K, T_0, C > 0$$

1.1. Time-independent solutions?

$\frac{dU}{dt} = 0 \qquad U = T(t) = T_0 + Ct$ which is not Time independent.

$\left(\frac{dU}{dt} = C \neq 0\right) \Leftarrow$ It is not even a solution.

There are not any time-independent solutions.

1.2

$$\frac{dU}{dt} + KU = KT_0 + KCt$$

Integrating factor $\mu \frac{dU}{dt} + K\mu U = \frac{d}{dt}[\mu U]$

$$\frac{d\mu}{dt} = K\mu \qquad \mu = e^{Kt}$$

$$\mu \frac{dU}{dt} + K\mu U = \mu K T_0 + \mu K C t$$

$$\frac{d}{dt}[e^{Kt} U] = K e^{Kt} T_0 + K e^{Kt} C t \quad \text{Integrating on both sides...}$$

$$e^{Kt} U = T_0 \int K e^{Kt} dt + C \int \underbrace{t K e^{Kt} dt}_{\text{we integrate by parts}} + D \quad \text{— constant}$$

$u = t \quad du = dt$

$dv = K e^{Kt} dt \quad v = e^{Kt}$

$$e^{Kt} U = T_0 e^{Kt} + C\left[t e^{Kt} - \int e^{Kt} dt\right] = T_0 e^{Kt} + C\left[t e^{Kt} - \frac{1}{K} e^{Kt}\right] + D$$

$$e^{Kt} U = e^{Kt} T_0 + e^{Kt} C\left(t - \frac{1}{K}\right) + D$$

$$\boxed{U = T_0 + C\left(t - \frac{1}{K}\right) + D e^{-Kt}} \quad \text{General solution}$$

$$U(0) = T_0 - \frac{C}{K} + D = U_0 \Rightarrow \quad D = U_0 + \frac{C}{K} - T_0$$

$$\boxed{U = T_0 + C\left(t - \frac{1}{K}\right) + \left(U_0 + \frac{C}{K} - T_0\right) e^{-Kt}} \quad \text{solution of IVP}$$

1.3 For $t \to \infty$, $\left(U_0 + \frac{C}{K} - T_0\right) e^{-Kt}$ decays exponentially fast.

For $t \gg \frac{1}{K}$ the initial value U_0 becomes irrelevant.

$U(t)$ grows linearly with Time with slope C, as $T(t)$ does.

$$U \approx T_0 + Ct - \frac{C}{K} = T(t) - \frac{C}{K}$$

If K is increased, the exponential decay becomes faster.

Additionally, the asymptotic behaviour becomes more similar to $T(t)$ as $\frac{C}{K}$ is smaller (but for long Times this difference becomes very small, anyway).

relatively speaking

2.1) $y_1 = 1+t$, $\dot{y}_1 = 1$, $\ddot{y}_1 = 0$

$t \cdot 0 - (1+t)\cdot 1 + 1 + t \overset{?}{=} 0$

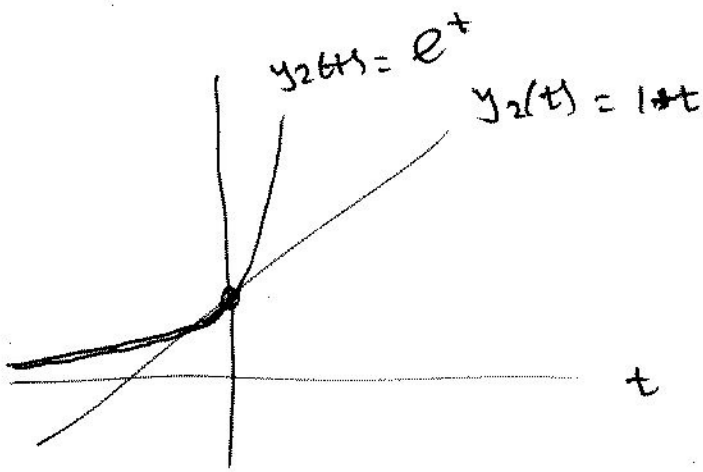
$0 = 0$

$y_2 = e^t$, $\dot{y}_2 = e^t$, $\ddot{y}_2 = e^t$

$t \cdot e^t - (1+t)e^t + e^t \overset{?}{=} 0$

$e^t(t - 1 - t + 1) = 0$

$e^t \cdot 0 = 0$



$\Rightarrow$ $y_1(t)$ is NOT proportional to $y_2(t)$, hence they are L.I.

2.2) $y_3(t) = v_1(t)\,y_1(t) + v_2(t)\,y_2(t) = v_1(t)(1+t) + v_2(t)\,e^t$

$\dot{y}_3 = \dot{v}_1(1+t) + v_1 + \dot{v}_2 e^t + v_2 e^t$

$\dot{v}_1(1+t) + \dot{v}_2 e^t = 0$

$\ddot{y}_3 = \dot{v}_1 + \dot{v}_2 e^t + v_2 e^t$

$t\ddot{y}_3 - (1+t)\dot{y}_3 + y_3 = t^2 e^{2t}$

$\rightarrow t\dot{v}_1 + \dot{v}_2 t e^t + v_2 t e^t - v_1 - t\dot{v}_1 - v_2 t e^t - v_2 e^t + v_1 + \dot{v}_1 t + v_2 e^t = t^2 e^{2t}$

$$\left.\begin{aligned} \dot{v}_1(1+t) + \dot{v}_2 e^t &= 0 \\ \dot{v}_1 t + \dot{v}_2 t e^t &= t^2 e^{2t} \end{aligned}\right\} \begin{aligned} &\rightarrow \quad \dot{v}_1 = \frac{-\dot{v}_2 e^t}{1+t} \\ &\rightarrow \quad \dot{v}_1 + \dot{v}_2 e^t = t e^{2t} \end{aligned}$$

$$\dot{v}_2 \underbrace{\left(\frac{-1}{1+t} + 1\right)}_{\frac{t}{1+t}} e^t = t e^{2t} \quad \Bigg\} \quad \dot{v}_2 = (1+t) e^t$$

$$\downarrow$$

$$v_2 = t e^t$$

$$\dot{v}_1 = \frac{-(1+t) e^t}{(1+t)} e^t = -e^{2t} \rightarrow v_1(t) = -\frac{1}{2} e^{2t}$$

$$\Rightarrow \boxed{y_3(t) = -\frac{1+t}{2} e^{2t} + t e^{2t} = \frac{t-1}{2} e^{2t}}$$