UC3M Fakemidterm, November 11, 2016

Advanced Mathematical Methods

Name: _____.

Exercise 1 Consider a large vat containing sugar water that is to be made into soft drinks. Suppose:

- The vat contains 100 gallons of liquid. Moreover, the amount flowing in is the same as the amount flowing out, so there are always 100 gallons in the vat.
- The vat is kept well mixed, so the sugar concentration is uniform throughout the vat.
- Sugar water containing 5 tablespoons of sugar per gallon enters the vat through pipe A at a rate of 2 gallons per minute.
- Sugar water containing 10 tablespoons of sugar per gallon enters the vat through pipe B at a rate of 1 gallon per minute.
- Sugar water leaves the vat through pipe C at a rate of 3 gallons per minute.
- Choose the total amount of sugar in the vat at time t measured in tablespoons as the dependent variable

1. Write a DE for the problem.
2. Solve the DE. How much sugar will be there for large times?
3. What can you say about uniqueness of the solution?

Exercise 2

Solve the initial-value problem

$$\frac{d^2y}{dt^2} + y = 1 + t\sin t.$$

with initial data $y(0) = 0$, $y'(0) = 0$. Do it using the methods of second order equations. Then transform it into a system and solve it using the methods for systems. Compare and discuss advantages and disadvantages of the two methods.

Exercise 3 Consider the system

$$\frac{dx}{dt} = -x^3,$$
$$\frac{dy}{dt} = -y + y^2.$$

1. Find and classify all the equilibrium points based on the linearisation.
2. Give the general (real) solution for each equilibrium point.
3. Draw the phase portrait with the different equilibrium points.
4. Will this model correctly the non-linear system?

Exercise 4 Proof the following theorem: If Φ is the fundamental matrix solution Φ to

$$\dot{\mathbf{\Phi}} = A\mathbf{\Phi}.$$

Then the solution to

$$\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{b},$$

with

$$\mathbf{x}(0) = \mathbf{x}_0,$$

is

$$\mathbf{x}(t) = \Phi(t)\Phi^{-1}(0)\mathbf{x}_0 + \int_0^t \Phi(t)\Phi^{-1}(\tau)\mathbf{b}(\tau)d\tau.$$