Universidad Carlos III de Madrid

Midterm exam
Advanced Mathematics

Aerospace Engineering
November 2016

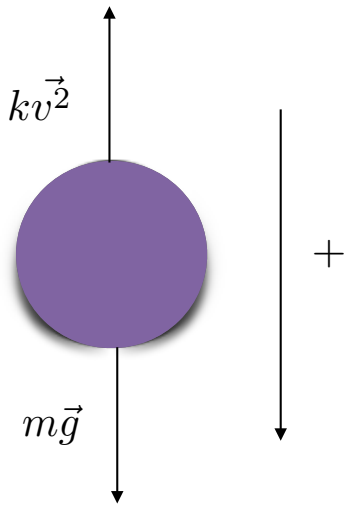
Name		Group	

Exercise 1 On a free-falling skydiver, apart from gravity, a force proportional to the square of the velocity is acting in the opposite direction of gravity.

a) (**0.75 pts**) Write down a differential equation, specifying the dependent and the independent variable and the parameters.

Sol: We consider the direction of falling as the positive x axis and define $v = \frac{dx}{dt}$. Gravity will increase the velocity in that direction and we have been told that the other force acts in the other direction, so that applying Newton' s law we have:

$$m\frac{dv}{dt} = mg - kv^2. \tag{1}$$



The dependent variable is the velocity v and the independent variable is time t. We consider the mass m of the sky diver to be constant. We also consider constant the acceleration rate

g which is approximately $9.81 m/s^2$. The proportionality factor of the other force we have denote by k which is a positive constant. The parameters are thus m, g and k. Of course it is also possible to choose the positive x axis in the opposite direction of falling so that $m\frac{dv}{dt} = -mg + kv^2$. We will stick to the other convention.

---

b) (**0.5 pts**) Make a qualitative analysis and obtain the terminal velocity without solving the differential equation.

Sol: We find the equilibrium point of our equation with

$$mg = kv^2. \tag{2}$$

So that we obtain

$$v = +\sqrt{\frac{mg}{k}}, \tag{3}$$

since the sky-diver is falling and we chose these direction to be positive. We see that if the skydiver is below this velocity

$$\frac{dv}{dt} > 0, \tag{4}$$

so that the velocity will increase until the terminal velocity is reached and if it is above

$$\frac{dv}{dt} < 0, \tag{5}$$

and the velocity will decrease.

---

c) (**0.75 pts**) Solve the differential equation and obtain the general solution assuming the sky-diver has zero initial velocity.

Sol: It can be solve by separation of variables and integration by reduction to a simple fractions:

$$m\frac{dv}{dt} = mg - kv^2 \rightarrow \frac{dv}{dt} = g - \frac{k}{m}v^2 = a^2 - b^2v^2$$

$$\int \frac{dv}{a^2 - b^2v^2} = \int dt$$

The integral is solve by reduction to simple fractions as follows

$$\int \frac{dv}{a^2 - b^2v^2} = \frac{-b}{2a}\int \frac{dv}{v + a/b} + \frac{b}{2a}\int \frac{dv}{v - a/b} = \frac{-b}{2a}\ln|v + a/b| + \frac{b}{2a}\ln|v + a/b|$$

$$\int \frac{dv}{a^2 - b^2 v^2} = \ln |\frac{v - a/b}{v + a/b}|^{\frac{b}{2a}}$$

Finally, we have:

$$\ln |\frac{v - a/b}{v + a/b}|^{\frac{b}{2a}} = t + K$$

where v can be expressed as:

$$v(t) = \frac{a}{b} \frac{1 - e^{\frac{2at}{b}}}{1 + e^{\frac{2at}{b}}}$$

Note that the constant have been obtained imposing the initial condition $v(0) = 0$.

---

d) (**0.5 pts**) Confirm the answer you obtained previously about the terminal velocity using the solution of the differential equation.

When t goes to infinity the obtained velocity matches with the already done qualitative analysis.

Exercise 2 Consider the initial-value problem

$$y'' - y' - 2y = 3 + 4\sin 2t + 2e^{-t}.$$

with initial data $y(0) = 0$, $y'(0) = 0$.

---

a) (**1 pts**) Solve it using the one of the methods of second order equations.

Sol: Substituting e^{rt} for y, we find that the characteristic equation is

$$r^2 - r - 2 = 0$$
$$(r - 2)(r + 1) = 0$$

Hence the homogeneous solution is

$$y = c_1 e^{-t} + c_2 e^{2t}$$

For the particular solution we guess

$$y_p = A + B\sin(2t) + C\cos(2t) + De^{-t}t$$

so

$$y_p' = 2B\cos(2t) - 2C\sin(2t) + De^{-t} - De^{-t}t$$
$$y_p'' = -4B\sin(2t) - 4C\cos(2t) - 2De^{-t} + De^{-t}t$$

Then, substituting into the differential equation we have

$$(-6B + 2C)\sin(2t) + (-6C - 2B)\cos(2t) - 3De^{-t} - 2A = 3 + 4\sin 2t + 2e^{-t}.$$

$$\begin{aligned} -6B + 2C &= 4 \\ -2A &= 3 \\ -6C - 2B &= 0 \\ -3D &= 2 \end{aligned}$$

solving the system we have $A = -\frac{3}{2}$, $B = -\frac{3}{5}$, $C = \frac{1}{5}$ and $D = -\frac{2}{3}$. Finally, we obtain

$$y_p = -\frac{3}{2} - \frac{3}{5}\sin(2t) + \frac{1}{5}\cos(2t) - \frac{2}{3}e^{-t}t$$

---

b) (**1 pts**) Transform it into a system and solve it using the methods for systems.

Sol: In order to transform it into a first order system, let us call

$$\frac{dy}{dx} = z$$

Then we obtain the following system

$$\begin{aligned} \frac{dy}{dx} &= x \\ \frac{dx}{dx} &= x + 2y + 3 + 4\sin(2t) + 2e^{-t} \end{aligned}$$

Analogously,

$$\begin{pmatrix} y' \\ x' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} y \\ x \end{pmatrix} + \begin{pmatrix} 0 \\ 3 + 4\sin(2t) + 2e^{-t} \end{pmatrix}$$

Computing the corresponding eigenvalues and eigenvectors, the homogeneous solution obtained is

$$\begin{pmatrix} y_h \\ x_h \end{pmatrix} = C_1 e^{2t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} + C_2 e^{-t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Thus, the fundamental matrix is

$$\Psi(t) = \begin{pmatrix} e^{2t} & -e^{-t} \\ -2e^{2t} & e^{-t} \end{pmatrix}$$

Then, the solution is given by $\begin{pmatrix} \mathrm{y(t)} \\ \mathrm{x(t)} \end{pmatrix} = \Psi(t)u(t)$, where $u(t)$ satisfies $\Psi(t)u'(t) = g(t)$

$$\begin{pmatrix} e^{2t} & -e^{-t} \\ -2e^{2t} & e^{-t} \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ 3 + 4\sin(2t) + 2e^{-t} \end{pmatrix}$$

solving by row reduction, we obtain $e^{3t}u_1' = u_2'$.
Finally, solving u_1', multiplying by e^{3t} in order to obtain u_2' we obtain our solution:

$$y(t) = \tfrac{1}{3}C_1 e^{-t}(e^{3t} - 1) + \tfrac{1}{3}C_2 e^{-t}(e^{3t} + 2) - \tfrac{1}{90}e^{-t}(60t + 135e^t + 54e^t \sin(2t) + 20) + \tfrac{1}{5}\cos(2t)$$

$$x(t) = \tfrac{1}{3}C_1 e^{-t}(2e^{3t} + 1) + \tfrac{2}{3}C_2 e^{-t}(e^{3t} - 1) - \tfrac{2}{45}e^{-t}(-15t + 9e^t \sin(2t) + 27e^t \cos(2t) + 10)$$

---

c) (**0.5 pts**) Compare and discuss advantages and disadvantages of the two methods.

Sol: Using the methods for second order equations is quicker to solve the problem, since one just has to solve a quadratic equation and one immediately knows the solution for the homogeneous equation. By the method of undetermined coefficients or variation of parameters one can finish the complete problem quickly. There are however three advantages using the methods for systems. One is that one obtains much more information since one has the eigenvectors which give you geometric information. We have seen based on the sign of the eigenvalues and on properties of the eigenvectors how to obtain a lot of different patterns. Second the methods for systems can be generalized easily to non-linear systems while this can be done only partially in the case of the other methods. And third, solving the problem as a system of first order differential equations with a numerical method implementation is always easier than the numerical implementation of higher order differential equations.

Exercise 3 Consider the system

$$\frac{dx}{dt} = -2x + y,$$
$$\frac{dy}{dt} = -y + x^2$$

Find the equilibrium points. Find the linearized systems, classify the equilibrium point and sketch the phase portraits.

a) (**0.75 pts**) Find and classify all the equilibrium points based on the linearization.

Sol: First, we find the equilibrium points from the following system of equations

$$-2x + y = 0 \quad -y + x^2 = 0$$

The equilibrium points are $p_1 = (0, 0)$ and $p_2 = (2, 4)$. We compute the Jacobian matrix which is

$$J = \begin{pmatrix} \text{-2} & 1 \\ \text{2x} & \text{-1} \end{pmatrix}$$

At the equilibrium point p_1 we have

$$J = \begin{pmatrix} \text{-2} & 1 \\ \text{2x} & \text{-1} \end{pmatrix}|_{(0,0)} = \begin{pmatrix} \text{-2} & 1 \\ 0 & \text{-1} \end{pmatrix}$$

Computing the eigenvalues we obtain $\lambda_1 = -1$ and $\lambda_2 = -2$. So we have a sink.

Analogously for p_2, we have

$$J = \begin{pmatrix} \text{-2} & 1 \\ \text{2x} & \text{-1} \end{pmatrix}|_{(2,4)} = \begin{pmatrix} \text{-2} & 1 \\ 4 & \text{-1} \end{pmatrix}$$

Computing the eigenvalues we obtain $\lambda = \frac{-3 \pm \sqrt{17}}{2}$. So we have a saddle.

---

b) (**0.75 pts**) Give the general (real) solution for each equilibrium point.

Sol: Computing the eigenvectors for the equilibrium point $p_1 = (0, 0)$, we obtain the following systems

For $\lambda_1 = -1$

$$\left(\begin{array}{cc|c} \text{-2 -}\lambda_1 & 1 & 0 \\ 0 & \text{-1-}\lambda_1 & 0 \end{array}\right) = \left(\begin{array}{cc|c} \text{-1} & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \rightarrow v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

For $\lambda_2 = -2$

$$\left(\begin{array}{cc|c} \text{-2 -}\lambda_2 & 1 & 0 \\ 0 & \text{-1-}\lambda_2 & 0 \end{array}\right) = \left(\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 1 & 0 \end{array}\right) \rightarrow v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Thus, we have the following solution:

$$y_1(t) = C_1 e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{-2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Computing the eigenvectors for the equilibrium point $p_2 = (2, 4)$, we obtain the following systems

For $\lambda_1 = \frac{-3+\sqrt{17}}{2}$

$$\left(\begin{array}{cc|c} \text{-2 -}\lambda_1 & 1 & 0 \\ 4 & \text{-1-}\lambda_1 & 0 \end{array}\right) = \left(\begin{array}{cc|c} \frac{-1-\sqrt{17}}{2} & 1 & 0 \\ 4 & \frac{1-\sqrt{17}}{2} & 0 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & \frac{1-\sqrt{17}}{8} & 0 \end{array}\right) \rightarrow v_1 = \begin{pmatrix} \frac{-1+\sqrt{17}}{8} \\ 1 \end{pmatrix}$$

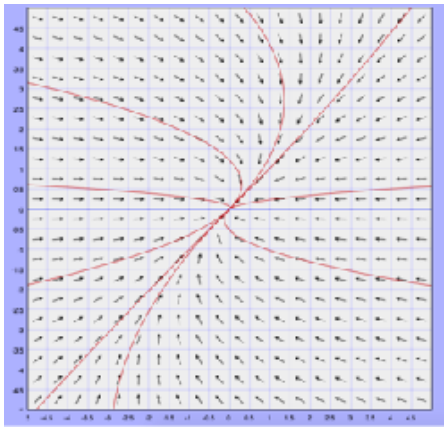
For $\lambda_2 = \frac{-3-\sqrt{17}}{2}$

$$\left(\begin{array}{cc|c} \text{-2 -}\lambda_2 & 1 & 0 \\ 4 & \text{-1-}\lambda_2 & 0 \end{array}\right) = \left(\begin{array}{cc|c} \frac{-1+\sqrt{17}}{2} & 1 & 0 \\ 4 & \frac{1+\sqrt{17}}{2} & 0 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & \frac{1+\sqrt{17}}{8} & 0 \end{array}\right) \rightarrow v_2 = \begin{pmatrix} \frac{-1-\sqrt{17}}{8} \\ 1 \end{pmatrix}$$

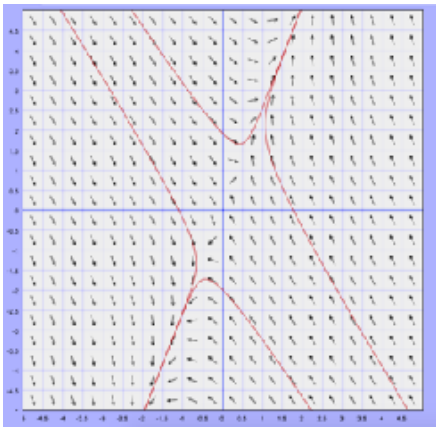
Thus, we have the following solution:

$$y_2(t) = C_1 \exp(\frac{-3+\sqrt{17}}{2}t) \begin{pmatrix} \frac{-1+\sqrt{17}}{8} \\ 1 \end{pmatrix} + C_2 \exp(\frac{-3-\sqrt{17}}{2}) \begin{pmatrix} \frac{-1-\sqrt{17}}{8} \\ 1 \end{pmatrix}$$

c) (**0.75 pts**) Draw the phase portrait with the different equilibrium points.



$y_1(t)$



$y_2(t)$

d) (**0.25 pts**) Will this model correctly the non-linear system?

Sol: Yes, it will model the non-linear system correctly, because we have seen that this is the case if none of the eigenvalues vanishes or has zero real part. Since we have a saddle and a sink which of course have non-vanishing real part eigenvalues, also for the non-linear case we will have the same qualitative behavior, we will also have the behavior of the saddle and the sink in the non-linear case.

Exercise 4 (**2.5 pts**) Proof Abel's theorem which says:

If y_1 and y_2 are solutions of the DE

$$y'' + p(t)y' + q(t)y = 0,$$

where p and q are continuous on an open interval. Then the Wronskian $W(y_1, y_2)$ is given by

$$W(y_1, y_2) = C \exp\left[-\int p(s)ds\right],$$

where C is a parameter that might depend on y_1 and y_2 but does not depend on t. Furthermore either W is zero for all $t \in I$ or never.

Sol: To prove Abel's theorem we start by noting that y_1 and y_2 satisfy

$$y_1'' + p(t)y_1' + q(t)y_1 = 0$$
$$y_2'' + p(t)y_2' + q(t)y_2 = 0$$

Multiplying the first equation by $-y_2$, the second by y_1, and adding the resulting equations, we obtain

$$(y_1y_2'' - y_1''y_2) + p(t)(y_1y_2' - y_1'y_2) = 0.$$

Letting $W(t) = W(y_1, y_2)(t)$, and noting that

$$W' = y_1y_2'' - y_1''y_2,$$

allows us to write

$$W' + pW = 0$$

which can be solved immediately since it is both first order linear equation and a separable equation

$$W(t) = C \exp\left[-\int p(s)ds\right],$$

where C is a constant. Since the exponential function is never zero, W(t) is not zero unless $c = 0$, in which case $W(x)$ is zero for all t, which completes the proof.