AEROSPACE ENGINEERING.
Advanced Mathematics. Midterm exam. November 16th, 2015.
Calculators, phones, tablets not allowed

Select and solve four out of the following five problems.

1. The partridge population near a hunting lodge is modeled by the ODE,

$$P' = (1 - P/1000)P - H,$$

where H is the harvesting rate (number of partridges taken per year).

   (a) How many licenses can be issued per year so that the partridge population has a chance of survival? Each hunter is allowed to shoot up to 20 partridges per year.

   (b) Suppose that N licenses are issued, where N is less than the maximal number found in the previous paragraph. What values of the initial partridge population lead to total extinction of the species? Explain.

Solution. (a) The equilibrium points are $P - P^2/1000 - H = 0$, *or* $P^2 - 1000P + 1000H = 0$. *This gives* $P_\pm = 500 \pm \sqrt{250000 - 1000H}$. *For* $H < 250$ *there are two EPs and none for* $H > 250$. *If* $H = 20N$, *the population is sustainable if* $N < 250/20 = 12.5$. *So at most 12 licenses can be issued per year. (b) For initial values* $P(0) < P_-$, $P' < 0$ *and the population goes extinct.* $P_- = 500 - 100\sqrt{25 - 2N}$.

2. Find the solution of the initial value problem

$$\begin{cases} y'' + 4y' + 4y = \sin t, \\ y(0) = 0,\ y'(0) = 0. \end{cases}$$

Solution. A particular solution of the ODE is the imaginary part of $y_p = e^{it}/(3 + 4i) = (3 - 4i)e^{it}/25$. *This is* $(3\sin t - 4\cos t)/25$. *As inserting* $y = e^{rt}$ *in the homogeneous equation gives* $r^2 + 4r + 4 = (r + 2)^2 = 0$, *the general solution of the ODE is*

$$y = (at + b)e^{-2t} + \frac{3}{25}\sin t - \frac{4}{25}\cos t.$$

Then the initial conditions yield $b = \frac{4}{25}$ *and* $a - 2b + \frac{3}{25} = 0$, *that gives* $a = \frac{1}{5}$. *The solution is*

$$y = \frac{5t + 4}{25}e^{-2t} + \frac{3}{25}\sin t - \frac{4}{25}\cos t.$$

3. Use Taylor expansions and the mean value theorem to calculate the truncation error of the Ralston method, consisting of replacing the IVP $u' = f(t,u)$, $u(t_0) = u_0$, by the scheme:

$$u_{j+1} = u_j + \frac{h}{4}\left[f(t_j,u_j) + 3f\left(t_j + \frac{2h}{3}, u_j + \frac{2h}{3}f(t_j,u_j)\right)\right], \quad u_0 = u(t_0),$$

with $t_{j+1} = t_j + h$ ($j = 0,1,\ldots,N-1$; $h = \frac{T-t_0}{N}$). What is the order of the method?

Solution.

$$\tau_{j+1} = \frac{u(t_j+h)-u(t_j)}{h} - \frac{1}{4}\left[f + 3f\left(t_j + \frac{2h}{3}, u(t_j) + \frac{2h}{3}f\right)\right], \quad f = f(t_j,u(t_j)).$$

Taylor expansion provides the result $\tau_{j+1} = \frac{h^2}{6}f_u(f_t + f_u f) + \ldots$, *so that the method is second order. Note:*

$$\frac{3}{4}f\left(t_j + \frac{2h}{3}, u(t_j) + \frac{2h}{3}f\right) = \frac{3}{4}f + \frac{h}{2}(f_t + f_u f) + \frac{h^2}{6}(f_{tt} + 2f_{tu}f + f_{uu}f^2) + \ldots$$

$$\frac{u(t_j+h)-u(t_j)}{h} = u' + \frac{h}{2}u'' + \frac{h^2}{6}u''' + \ldots = f + \frac{h}{2}\frac{d}{dt}f(t,u(t)) + \frac{h^2}{6}\frac{d^2}{dt^2}f(t,u(t)) + \ldots$$

$$= f + \frac{h}{2}(f_t + f_u f) + \frac{h^2}{6}[f_{tt} + 2f_{tu}f + f_{uu}f^2 + f_u(f_t + f_u f)] + \ldots$$

4. Consider the following system:

$$\begin{cases} x' = x^2 - y, \\ y' = x - y. \end{cases}$$

Find the equilibrium points and classify them by linearizing the system of equations about these EPs and analyzing the resulting eigenvalue problems.

Solution. The EPs solve the system: $x^2 - y = 0$ *and* $x - y = 0$, *which gives the two points:* $(0,0)$, $(1,1)$. *The linearization of the system about* (x_0,y_0) *yields the following coefficient matrix:*

$$\begin{pmatrix} 2x_0 & -1 \\ 1 & -1 \end{pmatrix} \Longrightarrow \begin{cases} \tau = 2x_0 - 1 \text{ (trace)} \\ \Delta = 1 - 2x_0 \text{ (determinant)} \end{cases}$$

The eigenvalues corresponding to the EPs and the type of the latter are: $\lambda = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$ *for the stable spiral point* $(0,0)$, $\Delta = -1 < 0$ *(two eigenvalues of opposite sign,* $\lambda = \frac{1}{2} \pm \frac{\sqrt{5}}{2}$*) for the saddle point* $(1,1)$.

5. Find the eigenvalues and eigenfunctions of the boundary value problem:

$$y'' - y' + \lambda y = 0, \quad y(0) = 0, \quad y(\pi) = 0.$$

Solution. If $y = e^{rx}$, the ODE gives $r^2 - r + \lambda = 0$, so that

$$r = \frac{1 \pm \sqrt{1 - 4\lambda}}{2} = \frac{1}{2} \pm i\sqrt{\lambda - \frac{1}{4}}.$$

Then $y(x) = e^{x/2}\sin(\gamma x)$ is a solution of the ODE that satisfies $y(0) = 0$ provided $\gamma = \sqrt{\lambda - \frac{1}{4}}$. $y(\pi) = 0$ implies that $\gamma = 1, 2, \ldots$, and the eigenvalues and eigenfunctions are

$$\lambda_n = n^2 + \frac{1}{4}, \quad y_n(x) = e^{x/2}\sin(nx), \quad n = 1, 2, \ldots.$$