AEROSPACE ENGINEERING.
Advanced Mathematics. Midterm exam. November 17th, 2014.
Calculators, phones, tablets not allowed

Select and solve four out of the following five problems.

1. Use Taylor expansions and the mean value theorem to calculate the truncation error of the modified Euler method (mid-point method), consisting of replacing the IVP

$$\frac{du}{dt} = f(t,u), \quad u(t_0) = u_0,$$

by the scheme:

$$u_{j+1} = u_j + h\, f\left(t_j + \frac{h}{2}, u_j + \frac{h}{2} f(t_j, u_j)\right), \quad u_0 = u(t_0),$$

with $t_{j+1} = t_j + h$ ($j = 0, 1, \ldots, N-1$; $h = \frac{T-t_0}{N}$). What is the order of the method?

Solution. By definition, the truncation error of the modified Euler scheme is

$$\tau_{j+1}(h) = \frac{u(t_j+h) - u(t_j)}{h} - f\left(u(t_j) + \frac{h}{2} f(t_j, u(t_j)), t_j + \frac{h}{2}\right).$$

We now use the Taylor theorem to expand $\tau_{j+1}(h)$ to order h^2:

$$\tau_{j+1}(h) = u'(t_j) + \frac{hu''(t_j)}{2} + \frac{h^2}{6}u'''(\xi_j) - f(u_j,t_j) - \frac{h}{2}[f_u(u_j,t_j)f(u_j,t_j) + f_t(u_j,t_j)]$$
$$+\frac{h^2}{8}[f_{uu}(u_j + \eta f(u_j,t_j), t_j+\eta)(f(u_j,t_j))^2 + 2f_{ut}(u_j+\eta f(u_j,t_j), t_j+\eta)f(u_j,t_j)$$
$$+f_{tt}(u_j+\eta f(u_j,t_j), t_j+\eta)],$$

where we have used the abbreviation $u_j = u(t_j)$, $t_j \le \xi_j \le t_j + h$, and $0 \le \eta \le h/2$. Using $u'(t_j) = f(u_j,t_j)$, $u''(t_j) = f_u(u_j,t_j)f(u_j,t_j) + f_t(u_j,t_j)$ and $u'''(\xi_j) = f_{uu}f^2 + 2f_{ut}f + f_{tt} + f_u f_t + f_u^2 f$, the terms of order 1 and h cancel and the leftover terms are of order h^2.

2. Consider the Lotka-Volterra predator-prey system:

$$\begin{cases} \frac{dx}{dt} = x - xy, \\ \frac{dy}{dt} = xy - y. \end{cases}$$

Find the equilibrium points and classify them by linearizing the Lotka-Volterra equations about these equilibrium points and solving the resulting linear ODEs.

Solution. The EPs satisfy $x(1-y)=0$ *and* $y(x-1)=0$. *Then there are two EPs:* $(0,0)$ *and* $(1,1)$. *The linearized ODEs about* $(0,0)$ *are* $\tilde{x}'=\tilde{x}$, $\tilde{y}'=-\tilde{y}$ *so that this EP is a saddle point whose separatrices are the coordinate axes (eigenvalues* ± 1, *solution* $\tilde{x}=ae^{t}$, $\tilde{y}=be^{-t}$*). The linearized ODEs about* $(1,1)$ *are* $\tilde{x}'=-\tilde{y}$, $\tilde{y}'=\tilde{x}$ *so that this EP is a center (eigenvalues* $\pm i$, *solution* $\tilde{x}=a\cos t+b\sin t$, $\tilde{y}=-\tilde{x}'=a\sin t-b\cos t$*).*

3. The trajectories of the Lotka-Volterra system, $y(x)$, satisfy the first order ODE

$$\frac{dy}{dx}=\frac{(x-1)y}{(1-y)x}.$$

Find the general solution that depends on an arbitrary constant.

Solution. By separation of variables, we find

$$\frac{1-y}{y}\,dy=\frac{x-1}{x}\,dx \Longrightarrow \int\left(\frac{1}{y}-1\right)dy=\int\left(1-\frac{1}{x}\right)dx+C.$$

Then,

$$\ln y-y=x-\ln x+C \Longleftrightarrow xye^{-xy}=K.$$

Thus $V(x,y)=\ln x+\ln y-y-x$ *is a first integral of the Lotka-Volterra system which implies that its EPs can only be either saddles or centers of the nonlinear system.*

4. Find the solution of the initial value problem

$$\begin{cases} y''+2y'+y=e^{-t}, \\ y(0)=1,\ y'(0)=0. \end{cases}$$

Solution. Firstly, we calculate that general solution of the linear homogeneous ODE. $y=e^{rx}$ *gives* $r^2+2r+1=0$, *so that* $r=-1$ *is a double root. Then* e^{-t} *and* te^{-t} *are independent solutions of the homogeneous ODE. A particular solution of the inhomogeneous ODE is* $y_p=At^2e^{-t}$. *We get* $e^{-t}=y_p''+2y_p'+y_p=2Ae^{-t}$, *so that* $A=\frac{1}{2}$. *Then the general solution of the inhomogeneous ODE is* $y=(\frac{1}{2}t^2+at+b)e^{-t}$. *The constants* a *and* b *follow from the initial conditions:* $y(0)=b=1$ *and*

$$y'(0)=a-1=0 \Longrightarrow a=1.$$

The solution of the IVP is

$$y(t)=\left(\frac{1}{2}t^2+t+1\right)e^{-t}.$$

Another way to find the solution of the inhomogeneous ODE is by reduction of order. Inserting $y=e^{-t}u(t)$ *in the ODE, we get* $u''e^{-t}=e^{-t}$, *i.e.,* $u''=1$. *Then* $u=\frac{t^2}{2}+at+b$ *which produces the same solution as found before.*

5. Find the eigenvalues and eigenfunctions of the boundary value problem: $y'' + \frac{9}{4}y + \lambda y = 0$, $y'(0) = 0$, $y(\pi) = 0$. Expand the function x in terms of the eigenfunctions of the previous eigenvalue problem.

Solution. Setting $y = e^{rx}$, *we get* $r^2 + \frac{9}{4} + \lambda = 0$, *that gives* $r = \pm i\sqrt{\lambda + \frac{9}{4}}$. *Then* $u = c_1 \cos(\sqrt{\lambda + \frac{9}{4}}x) + c_2 \sin(\sqrt{\lambda + \frac{9}{4}}x)$. *The boundary conditions yield* $c_2 = 0$ *and* $\cos\left(\pi\sqrt{\lambda + \frac{9}{4}}\right) = 0$. *Therefore* $\sqrt{\lambda + \frac{9}{4}}\pi = (n - \frac{1}{2})\pi$ *with* $n = 1, 2, \ldots$ *and we have obtained the eigenvalues* $\lambda_n = (n - \frac{1}{2})^2 - \frac{9}{4} = n^2 - n - 2$, $n = 1, 2, \ldots$ *(*$\lambda_n = -2, 0, 4, 10, \ldots$*), and the eigenfunctions* $\phi_n(x) = \cos[(n - \frac{1}{2})x]$.
The Fourier coefficients a_n *of the function* $f(x) = x$ *are*

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^\pi x\cos[(n-\frac{1}{2})x]\,dx = \frac{4}{(2n-1)\pi}\Big\{x\sin[(n-\frac{1}{2})x]|_0^\pi - \int_0^\pi \sin[(n-\frac{1}{2})x]\,dx\Big\} \\ &= -\frac{4}{(2n-1)^2\pi}[(-1)^n\pi(2n-1)+2]. \end{aligned}$$

Then

$$x = -\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{2+(-1)^n\pi(2n-1)}{(2n-1)^2}\cos\frac{(2n-1)x}{2}.$$