AEROSPACE ENGINEERING.
Advanced Mathematics. Midterm exam. November 18th, 2013.
Calculators, phones, tablets not allowed

Select and solve four out of the following five problems.

1. The number of compulsive gamblers in the Madrid region increases at a rate proportional to their current population $y(t)$. Their growth rate is proportional to $1-y/C$, where the carrying capacity C of potential Eurovegas customers is three million. In an ideal situation, when there are very few gamblers compared to C, their number duplicates every three months in the absence of any other factors. Once the casinos are fully operational, the ordinary action of local parishes, psychotherapists and other philanthropic types may wean a thousand suckers off gambling per month. With an estimated ten thousand initial inveterate gamblers, these efforts come to nothing. Calculate the minimum number of repentant sinners they need to produce every month (possibly helped by a change in the tax code) to drive gambling to extinction.

   Solution. The gambling population y evolves according to the ODE:

$$\frac{dy}{dt} = ry\left(1 - \frac{y}{C}\right) - K,$$

   where K is the number of repentant gamblers per month. In the absence of any other factors, $dy/dt \approx ry$ gives $y(t) = y_0 e^{rt}$. Duplication every three months mean $2y_0 = y_0 e^{3r}$ or $r = \frac{1}{3}\ln 2$ per month. With $K = C/3000$ (1000 gamblers per month), the gambling population has two real-valued equilibria ($dy/dt = 0$) $y_\pm = \frac{C}{2} \pm \sqrt{\frac{C^2}{4} - \frac{CK}{r}}$. Notice that $dy/dt < 0$ for $y < y_-$ and $y > y_+$, whereas $dy/dt > 0$ for $y_- < y < y_+$. For all initial conditions $y_0 > y_-$, the number of gamblers tends to the other equilibrium point y_+, whereas they are driven to extinction in a virtuous society where $y_0 < y_-$. In our example, $y_- \approx 4334$ which is certainly a small number (close to $K/r \approx 4328$) as compared to $y_0 = 10,000$. To eradicate the gambling vice, the righteous crowd should achieve $K_c = rC/4 \approx 173,287$ conversions per month or more, so that $y_+ = y_- = C/2$ (corresponding to $f(y) = f'(y) = 0$ with the maximum of the parabola $f(y) = ry(1 - y/C) - K = \frac{dy}{dt}$ equal to 0) and $dy/dt < 0$ for any larger value $K > K_c$.

2. Classify the fixed point of the following linear system:

$$\begin{cases} \frac{dx}{dt} = 4x - y, \\ \frac{dy}{dt} = 2x + y. \end{cases}$$

Find the general solution of the system and solve it for the initial condition $(x_0, y_0) = (3, 4)$.

Solution. The characteristic equation is $\lambda^2 - 5\lambda + 6 = 0$ *and the eigenvalues are 2, with eigenvector* $(1, 2)$*, and 3, with eigenvector* $(1, 1)$*. The origin is an unstable node. The general solution of the system is*

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t}.$$

For the given initial condition, we find $c_1 = 1$*,* $c_2 = 2$*, hence* $x(t) = e^{2t} + 2e^{3t}$*,* $y(t) = 2e^{2t} + 2e^{3t}$*.*

3. Find the general solution of the system of ODEs

$$\begin{cases} \frac{dx}{dt} = -3x + 4y + \cos t, \\ \frac{dy}{dt} = -2x + 3y + 1. \end{cases}$$

Solution. We know that the eigenvalues are ± 1*, with eigenvectors* $(1, 1)$ *and* $(2, 1)$*, respectively. The fundamental matrix is therefore*

$$\underline{\underline{\Psi}} = \begin{pmatrix} e^t & 2e^{-t} \\ e^t & e^{-t} \end{pmatrix}.$$

Its inverse is

$$\underline{\underline{\Psi}}^{-1} = \begin{pmatrix} -e^{-t} & 2e^{-t} \\ e^t & -e^t \end{pmatrix}.$$

Then

$$\int \underline{\underline{\Psi}}^{-1} \underline{F}(t) dt = \int \begin{pmatrix} e^{-t}(2 - \cos t) \\ e^t(\cos t - 1) \end{pmatrix} dt = \begin{pmatrix} -e^{-t}[2 + \frac{1}{2}(\sin t - \cos t)] \\ e^t[\frac{1}{2}(\cos t + \sin t) - 1] \end{pmatrix}.$$

Using now

$$\underline{x}(t) = \underline{\underline{\Psi}}(t) \int \underline{\underline{\Psi}}^{-1} \underline{F}(t) dt = \begin{pmatrix} -4 + \frac{1}{2}\sin t + \frac{3}{2}\cos t \\ -3 + \cos t \end{pmatrix},$$

we find a particular solution to which we add the general solution of the homogeneous system:

$$\underline{x}(t) = \underline{\underline{\Psi}}(t) \int \underline{\underline{\Psi}}^{-1} \underline{F}(t) dt = \begin{pmatrix} -4 + \frac{1}{2}\sin t + \frac{3}{2}\cos t + c_1 e^t + 2c_2 e^{-t} \\ -3 + \cos t + c_1 e^t + c_2 e^{-t} \end{pmatrix}.$$

The method of undetermined coefficients would give $\underline{x}_p = \underline{a} + \underline{b}\cos t + \underline{c}\sin t$ *which, inserted in the system produces* $-\underline{\underline{A}}\underline{a} = (0, 1)$*,* $\underline{c} - \underline{\underline{A}}\underline{b} = (1, 0)$ *and* $-\underline{b} = \underline{\underline{A}}\underline{c}$*. Since* $\underline{\underline{A}}^2 = \underline{\underline{I}}$*,* $\underline{a} = -\underline{\underline{A}}(0, 1) = (-4, -3)$ *and* $\underline{c} = -\underline{\underline{A}}\underline{b}$*. Then* $2\underline{c} = (1, 0)$ *and* $\underline{c} = \frac{1}{2}(1, 0)$*,* $\underline{b} = -\frac{1}{2}\underline{\underline{A}}(1, 0) = (\frac{3}{2}, 1)$*. This yields the same particular solution as before.*

4. Find the eigenvalues and eigenfunctions of the boundary value problem: $u'' + 4u' + 3u + \lambda u = 0$, $u(0) = 0 = u(1)$. Write it as a Sturm-Liouville eigenvalue problem: $[p(x)\phi']' + q(x)\phi + \lambda\sigma(x)\phi = 0$, determine p, q and σ and write the orthogonality condition for the eigenfunctions.

   Solution. Setting $u = e^{rx}$, we get $r^2 + 4r + \lambda + 3 = 0$, that gives $r = -2 \pm i\sqrt{\lambda - 1}$. Then $u = e^{-2x}[c_1 \cos(\sqrt{\lambda - 1}x) + c_2 \sin(\sqrt{\lambda - 1}x)]$. The boundary conditions yield $c_1 = 0$ and $\sin\sqrt{\lambda - 1} = 0$. Therefore ($\sqrt{\lambda - 1} = n\pi$ with $n = 1, 2, \ldots$ and we have obtained the eigenvalues $\lambda_n = 1 + n^2\pi^2$, $n = 1, 2, \ldots$ and the eigenfunctions $\phi_n(x) = e^{-2x}\sin(n\pi x)$. To transform into a Sturm-Liouville ODE, we multiply the original equation by e^{4x}, thereby obtaining $(e^{4x}u')' + (\lambda + 3)e^{4x}u = 0$. We identify: $p(x) = e^{4x}$, $q(x) = 3e^{4x}$ and $\sigma(x) = e^{4x}$. The orthogonality condition for the eigenfunctions is $\int_0^1 e^{4x}\phi_m(x)\phi_n(x)\,dx = 0$ (i.e., $\int_0^1 \sin m\pi x \sin n\pi x\,dx = 0$) if $m \neq n$.

5. Find the Fourier sine series of $f(x) = 1 - x$ for $0 < x < 1$. What is the Fourier cosine series of $f'(x)$ with the same period?

   Solution. The Fourier coefficients are

$$b_n = 2\int_0^1 (1 - x)\sin n\pi x\,dx = -\frac{2}{n\pi}(1 - x)\cos n\pi x|_0^1 - \frac{2}{n\pi}\int_0^1 \cos n\pi x\,dx = \frac{2}{n\pi}.$$

   The Fourier sine series is therefore

$$\sum_{n=1}^{\infty} \frac{2}{n\pi}\sin n\pi x.$$

   The odd extension of $f(x)$ has a discontinuity at $x = 0$ therefore it is not possible to differentiate the previous series term by term. However, $f'(x) = -1$ in $(0, 1)$ has the even extension $f'(x) = -1$ to $(-1, 0)$ and the corresponding Fourier series is simply -1.