AEROSPACE ENGINEERING.
Advanced Mathematics. Midterm exam. October 25th, 2012.

Select and solve four out of the following five problems.

1. A spherical rain drop falls in the gravitational field starting from rest. Provided the drop picks up water vapor at a rate proportional to its surface and has zero initial radius, prove that the drop falls with constant acceleration and find it. Hint: Use that the force is the time derivative of the linear momentum (mass times velocity).

   Solution. We have $d(mv)/dt = -mg$ *and therefore*

   $$\frac{dv}{dt} = -g - v\frac{d}{dt}\ln m = -g - 3v\frac{d}{dt}\ln r,$$

   because $m = \frac{4}{3}\rho\pi r^3$. *We also have*

   $$\frac{dm}{dt} = k(4\pi r^2) \Longrightarrow \frac{dr}{dt} = \frac{k}{\rho}.$$

   This gives $r(t) = \frac{k}{\rho}t$ *because* $r(0) = 0$. *Then*

   $$\frac{dv}{dt} = -g - \frac{3v}{t}.$$

   Multiplying by the integrating factor t^3, *we obtain*

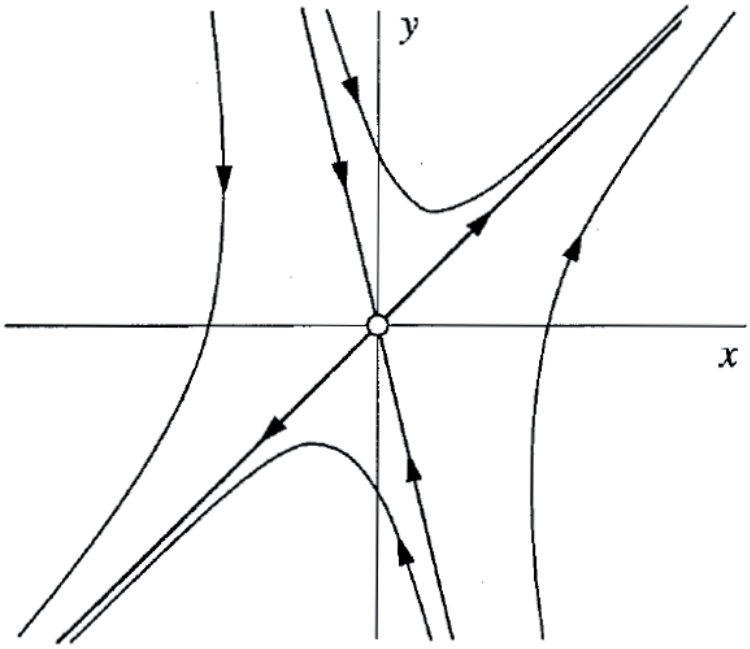
   $$t^3\frac{dv}{dt} + 3t^2v = \frac{d}{dt}(t^3v) = -gt^3.$$

   Integrating and using the initial condition, we get $t^3v = -gt^4/4$, *from* $v(t) = -gt/4$. *Then*

   $$\frac{dv}{dt} = -\frac{g}{4},$$

   which proves that the drop falls with constant acceleration $g/4$.

2. Find all functions $u(t)$, such that $tu'' - (1+t)u' + u = 0$. Hint. Try reduction of order if you can guess one solution.



Solution. By inspection, two solutions are $u_1 = 1 + t$ *and* $u_2 = e^t$. *They are independent and therefore the general solution is* $u(t) = c_1(1+t) + c_2e^t$. *If you can guess only one of them, for example,* u_2, *try* $u = u_2v$. *This yields* $tv'' + (t-1)v' = 0$, *i.e.,* $v''/v' = (1-t)/t$ *that gives* $\ln v' = \ln t - t + \ln c_2$. *Then* $v' = c_2te^{-t}$ *and* $v = -c_2(t+1)e^{-t} + c_1$, *and therefore* $u = c_1e^t - c_2(t+1)$ *as before.*

3. Plot the phase portrait and classify the fixed point of the following linear system:

$$\begin{cases} \frac{dx}{dt} = x + y, \\ \frac{dy}{dt} = 4x - 2y. \end{cases}$$

If the eigenvectors are real, indicate them in your sketch.

Solution. The characteristic equation is $\lambda^2 + \lambda - 6 = 0$ *and the eigenvalues are 2, with eigenvector* $(1, 1)$, *and -3, with eigenvector* $(1, -4)$. *The origin is a saddle point and the phase portrait can be drawn depicting the separatrices which are straight lines with directions given by the eigenvectors. See the figure where the arrows indicate the direction of increasing* t.

4. Find the general solution of the system of ODEs

$$\begin{cases} \frac{dx}{dt} = x + y + e^t, \\ \frac{dy}{dt} = 4x - 2y + 1. \end{cases}$$

Solution. We know that the eigenvalues are 2, with eigenvector $(1, 1)$, *and -3, with eigenvector* $(1, -4)$. *The fundamental matrix is therefore*

$$\underline{\underline{\Psi}} = \begin{pmatrix} e^{-3t} & e^{2t} \\ -4e^{-3t} & e^{2t} \end{pmatrix}$$

Its inverse is

$$\underline{\underline{\Psi}}^{-1} = \begin{pmatrix} e^{-3t} & e^{2t} \\ -4e^{-3t} & e^{2t} \end{pmatrix}$$

Then

$$\int \underline{\underline{\Psi}}^{-1}\underline{F}(t)dt = \frac{1}{5}\int \begin{pmatrix} e^{3t} & -e^{3t} \\ 4e^{-2t} & e^{-2t} \end{pmatrix}\begin{pmatrix} e^{t} \\ 1 \end{pmatrix} dt = \frac{1}{5}\int \begin{pmatrix} e^{4t}-e^{3t} \\ 4e^{-t}+e^{-2t} \end{pmatrix} dt.$$

Using now $\underline{x}(t) = \underline{\underline{\Psi}}(t)\int \underline{\underline{\Psi}}^{-1}\underline{F}(t)dt$, *we find a particular solution to which we add the general solution of the homogeneous system:*

$$\underline{x}(t) = \begin{pmatrix} -\frac{3}{4}e^{t} - \frac{1}{6} \\ -e^{t} + \frac{1}{6} \end{pmatrix} + \underline{\underline{\Psi}}\,\underline{c}.$$

The method of undetermined coefficients would give $\underline{x}_p = \underline{a}e^t + \underline{b}$ *which, inserted in the system produces* $(\underline{a} - \underline{\underline{A}}\underline{a} = (1,0)$ *and* $-\underline{\underline{A}}\underline{b} = (0,1)$. *Then* $\underline{a} = (\underline{\underline{I}} - \underline{\underline{A}})^{-1}(1,0) = ((-\frac{3}{4}, -1)$, $\underline{b} = -\underline{\underline{A}}^{-1}(0,1) = \frac{1}{6}(-1,1)$, *and this yields the same particular solution as before.*

5. Find the general solution of the ODE: $u'' + 4u' + 3u = \sin t$. Then determine the arbitrary constants so that $u(0) = 0$ and $u'(0) = 0$.

Solution. The homogeneous ODE has solution $u_h = c_1e^{-t} + c_2e^{-3t}$. *A particular solution of the ODE has the form:* $u_p = a\cos t + b\sin t$. *Then* $u_p'' + 4u_p' + 3u_p = (2a + 4b)\cos t + (2b - 4a)\sin t = \sin t$, *so that* $2b - 4a = 1$, $2a + 4b = 0$. *Then* $a = -1/5$, $b = 1/10$. *The general solution of the inhomogeneous ODE is*

$$u(t) = -\frac{1}{5}\cos t + \frac{1}{10}\sin t + c_1e^{-t} + c_2e^{-3t}.$$

We determine the constants c_1, c_2 *by inserting this solution in the initial conditions:*

$$c_1 + c_2 = \frac{1}{5}, \quad c_1 + 3c_2 = \frac{1}{10}.$$

$$c_1 = \frac{1}{4}, \quad c_2 = -\frac{1}{20}.$$

The solution of the IVP is

$$u(t) = -\frac{1}{5}\cos t + \frac{1}{10}\sin t + \frac{1}{4}e^{-t} - \frac{1}{20}e^{-3t}.$$