AEROSPACE ENGINEERING.
Advanced Mathematics. Midterm exam. October 26th, 2011.

Select and solve four out of the following five problems.

1. One day it started snowing at a heavy and steady rate. Three identical snowplows started out at noon, 1 p.m. and 2 p.m. from the same place, cleaning the same road, and all collided at the same time. What time did it start snowing? Hint: The constant speed of the snowplow is inversely proportional to the height of the snow. At a given point, the snow starts accumulating from zero height immediately after one snowplow has passed through that point.

   Solution. The height of the snow is $h = r't$, counted from the time it started snowing. The speed of the snowplow is then $dx/dt = \alpha/t$, where α is a positive constant. The solution of this problem satisfying $x(t_0) = 0$ (t_0 corresponds to 12 am) is $x(t) = \alpha \ln(t/t_0)$. Then the time at which this snowplow was at x was

$$t_1(x) = t_0 e^{x/\alpha}.$$

   After the first snowplow has passed through x (but before the second snowplow arrives at x), the snow height at a later time t is proportional to $t - t_1(x)$. Thus the second snowplow has a velocity $dx/dt = \alpha/[t - t_1(x)]$, and an initial condition $x(t_0 + 1) = 0$. That ODE can be written as $dt/dx = (t - t_0 e^{x/\alpha})/\alpha$ (with initial condition $t(0) = t_0 + 1$), which gives $d(te^{-x/\alpha})/dx = -t_0/\alpha$. Then the solution of the corresponding IVP is

$$t = t_2(x) = \left(t_0 + 1 - \frac{t_0 x}{\alpha}\right) e^{x/\alpha}.$$

   This is the time $t_2(x)$ at which the second snowplow goes through x. The third snowplow has a velocity $dx/dt = \alpha/[t - t_2(x)] = \alpha/[t - (t_0 + 1 - t_0 x/\alpha)e^{x/\alpha}]$ and it satisfies $x(t_0 + 2) = 0$. As before, we rewrite the ODE as $dt/dx = [t - (t_0 + 1 - t_0 x/\alpha)e^{x/\alpha}]/\alpha$, which gives $d(te^{-x/\alpha})/dx = -(t_0 + 1)/\alpha + t_0 x/\alpha^2$. Taking into account that $t(0) = t_0 + 2$ for the third snowplow, the solution of the IVP is

$$t = t_3(x) = \left[t_0 + 2 - (1 + t_0)\frac{x}{\alpha} + \frac{t_0 x^2}{2\alpha^2}\right] e^{x/\alpha}.$$

At time $t = t_c > t_0 + 2$ all the snowplows are at the same point x, so that $t_1(x) = t_2(x) = t_3(x)$. $t_1(x) = t_2(x)$ yields $t_0 = t_0 + 1 - t_0 x/\alpha$, which implies $x = \alpha/t_0$. $t_2(x) = t_3(x)$ yields $t_0 + 1 - t_0 x/\alpha = t_0 + 2 - (1 + t_0)x/\alpha + t_0 x^2/(2\alpha^2)$ and therefore $0 = 1 - x/\alpha + t_0 x^2/(2\alpha^2)$, thereby producing $0 = 1 - t_0^{-1} + 1/(2t_0)$. Then $t_0 = 1/2$ and it started snowing at 11:30 a.m.

2. Use Taylor expansions and the mean value theorem to calculate the truncation error of the modified Euler method (mid-point method), consisting of replacing the IVP

$$\frac{du}{dt} = f(u,t), \quad u(t_0) = u_0,$$

by the scheme:

$$w_{j+1} = w_j + h\, f\left(w_j + \frac{h}{2} f(w_j, t_j), t_j + \frac{h}{2}\right), \quad w_0 = u_0,$$

with $t_{j+1} = t_j + h$ ($j = 0, 1, \ldots, N-1$; $h = \frac{T - t_0}{N}$). What is the order of the method?

Solution. By definition, the truncation error of the mid-point scheme is

$$\tau_{j+1}(h) = \frac{u(t_j + h) - u(t_j)}{h} - f\left(u(t_j) + \frac{h}{2} f(t_j, u(t_j)), t_j + \frac{h}{2}\right).$$

We now use the Taylor theorem to expand $\tau_{j+1}(h)$ to order h^2:

$$u(t_j + h) = u(t_j) + h u'(t_j) + \frac{h^2}{2} u''(t_j) + \frac{h^3}{6} u'''(\xi_j),$$

$$f\left(u(t_j) + \frac{h}{2} f(t_j, u(t_j)), t_j + \frac{h}{2}\right) = f(u(t_j), t_j) + h \frac{d}{dh}\, f\left(u(t_j) + \frac{h}{2} f(t_j, u(t_j)), t_j + \frac{h}{2}\right)\Big|_{h=0} + \frac{h^2}{2}\, \frac{d^2}{dh^2}\, f\left(u(t_j) + \frac{h}{2} f(t_j, u(t_j)), t_j + \frac{h}{2}\right)\Big|_{h=2\eta}.$$

Using the chain rule, we find

$$u''(t) = \frac{d}{dt} f(u(t), t) = f_u(u(t), t) u'(t) + f_t(u(t), t) = f_u(u(t), t) f(u(t), t) + f_t(u(t), t),$$

and

$$\frac{d}{dh}\, f\left(u(t_j) + \frac{h}{2} f(t_j, u(t_j)), t_j + \frac{h}{2}\right)\Big|_{h=0} = \frac{1}{2} f_u(u(t), t) f(u(t), t) + \frac{1}{2} f_t(u(t), t),$$

with similar formulas for the second derivative. Insertion in the truncation error produces

$$\tau_{j+1}(h) = u'(t_j) + \frac{h u''(t_j)}{2} + \frac{h^2}{6} u'''(\xi_j) - f(u_j, t_j) - \frac{h}{2}[f_u(u_j, t_j) f(u_j, t_j) + f_t(u_j, t_j)] + \frac{h^2}{8}[f_{uu}(u_j + \eta f(u_j, t_j), t_j + \eta)(f(u_j, t_j))^2 + 2 f_{ut}(u_j + \eta f(u_j, t_j), t_j + \eta) f(u_j, t_j) + f_{tt}(u_j + \eta f(u_j, t_j), t_j + \eta)],$$

where we have used the abbreviation $u_j = u(t_j)$, and, according to the generalized mean value theorem, ξ_j and η are some numbers in the intervals $t_j \le \xi_j \le t_j + h$, and $0 \le \eta \le h/2$, respectively. Using $u'(t_j) = f(u_j, t_j)$ and the previous formula, $u''(t_j) = f_u(u_j, t_j)f(u_j, t_j) + f_t(u_j, t_j)$, the terms of order 1 and h cancel and the leftover terms in the truncation error have the form h^2Q, where Q contains first and second partial derivatives of f. Assuming that all second partial derivatives exist and are continuous, Q is a continuous function of t in the interval $t_0 \le t \le t_{max}$, and therefore $|\tau_{j+1}(h)| \le Ch^2$, where $C = max_{[t_0,t_{max}]}|Q|$. Thus the modified Euler scheme is a second-order method.

3. Plot the phase portrait and classify the fixed point of the following linear system:

$$\begin{cases} \frac{dx}{dt} = -3x + 4y, \\ \frac{dy}{dt} = -2x + 3y. \end{cases}$$

If the eigenvectors are real, indicate them in your sketch.

Solution. The coefficient matrix has zero trace and determinant -1. Its eigenvalues are ± 1 and the corresponding eigenvectors are

$$\underline{\Phi}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \underline{\Phi}_{-1} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

The origin is a saddle node and its separatrices have slopes 1 (unstable separatrix) and 1/2 (stable separatrix). With these data, it is immediate to draw the phase portrait.

4. (a) By reduction of order, find the general solution of $t^2u'' - 4tu' + 6u = t^4 \sin t$ after observing that $u_1 = t^2$ is a solution of the associated homogeneous equation. (b) Use the result to solve the previous ODE with boundary conditions $u(-\pi) = 0$, $u(\pi) = 2$.

Solution. (a) We set $u(t) = t^2v(t)$ and substitute in the ODE thereby getting $t^4v'' + 4t^3v' - 4t^3v' = t^4 \sin t$, i.e., $v'' = \sin t$ that produces (after integrating twice): $v(t) = -\sin t + at + b$, where a and b are arbitrary numbers. Then the general solution of the ODE is $u(t) = -t^2 \sin t + at^3 + bt^2$. (b) $0 = u(-\pi) = \pi^2(-a\pi + b)$, $2 = u(\pi) = \pi^2(a\pi + b)$. Then $a = 1/\pi^3$, $b = 1/\pi^2$. The solution of the BVP is $u(t) = -t^2 \sin t + (t/\pi)^3 + (t/\pi)^2$.

5. Find the general solution of the ODE

$$\frac{d^2u}{dt^2} + \frac{du}{dt} - 2u = \cos^2 t.$$

Reminder: $\cos^2 t = \frac{1+\cos(2t)}{2}$.

Solution. Since $\cos^2 t = (1+\cos 2t)/2$, the method of undetermined coefficients would give $u_p = A + B\cos 2t + C \sin 2t$ which, inserted in the ODE produces $u_p'' + u_p' - 2u_p =$

$(2C-6B)\cos 2t-(6C+2B)\sin 2t-2A = 1/2(1+\cos 2t)$. *Then* $A = -1/4$, $B = -3C$ *and* $2C - 6B = 1/2$, *so that* $C = 1/40$, $B = -3/40$*:*

$$u_p(t) = -\frac{1}{4} - \frac{3}{40}\cos 2t + \frac{1}{40}\sin 2t.$$

The solution of the homogeneous ODE $u_h = e^{\lambda t}$ *gives* $\lambda^2 + \lambda - 2 = 0$, *and therefore* $\lambda = 1, -2$. *Then the general solution of the ODE is*

$$u(t) = -\frac{1}{4} - \frac{3}{40}\cos 2t + \frac{1}{40}\sin 2t + c_1 e^t + c_2 e^{-2t}.$$

A faster method consists of solving the ODE $u'' + u' - 2u = 1/2 + e^{i2t}/2$ *assuming* $U = A + De^{i2t}$ *and then taking the real part, so that* $u_p(t) = Re\, U(t)$. *We again find* $A = -1/4$ *and* $(-4 + 2i - 2)D = 1/2$, *so that* $D = \frac{1}{4(i-3)}$. *Then we obtain* $u_p = -1/4 + \frac{1}{4}Re(-3 - i)e^{i2t}/10 = \frac{1}{40}[\sin(2t) - 3\cos(2t)] - 1/4$, *which is the same particular solution as before.*
Variation of parameters is more laborious. Two independent solutions of the homogeneous problem are $u_1 = e^t$ *and* $u_2 = e^{-2t}$ *with Wronskian* $W(t) = -3e^{-t}$. *Variation of parameters with* $u = v_1 e^t + v_2 e^{-2t}$ *gives* $v_1' e^t + v_2' e^{-2t} = 0$, *so that* $u' = e^t v_1 - 2e^{-2t} v_2$. *One more differential produces* $u'' = e^t(v_1' + v_1) - 2e^{-2t}(v_2' - 2v_2) = 3e^t v_1' + e^t v_1 + 4e^{-2t} v_2$. *Substituting into the ODE, we get* $\cos^2 t = (1 + \cos 2t)/2 = 3e^t v_1'$, *from which* $v_1' = e^{-t}(1 + \cos 2t)/6$. *Then*

$$v_1 = -\frac{e^{-t}}{6}\left(1 + \frac{1}{5}\cos 2t - \frac{2}{5}\sin 2t\right) + c_1,$$

where we have used the integral $J(-1)$, *defined through*

$$J(a) = \int e^{at}\cos(2t)dt = Re\int e^{(a+2i)t}dt = Re\frac{e^{(a+2i)t}}{a+2i} = e^{at}\frac{a\cos(2t) + 2\sin(2t)}{a^2+4}.$$

Now $v_2' = -v_1' e^{3t} = -e^{2t}(1 + \cos 2t)/6$, *which integrated using* $J(2)$ *produces*

$$v_2 = -\frac{e^{2t}}{24}(2 + \cos 2t + \sin 2t) + c_2.$$

Then, after some algebra, $u(t) = v_1(t)e^t + v_2(t)e^{-2t}$ *yields the same solution as before.*