AEROSPACE ENGINEERING.
Advanced Mathematics. Midterm exam. October 26th, 2011.

Select and solve four out of the following five problems.

1. One day it started snowing at a heavy and steady rate. Three identical snowplows started out at noon, 1 p.m. and 2 p.m. from the same place, cleaning the same road, and all collided at the same time. What time did it start snowing? Hint: The constant speed of the snowplow is inversely proportional to the height of the snow. At a given point, the snow starts accumulating from zero height immediately after one snowplow has passed through that point.

   Solution. The height of the snow is $h = r't$, counted from the time it started snowing. The speed of the snowplow is then $dx/dt = \alpha/t$, where α is a positive constant. The solution of this problem satisfying $x(t_0) = 0$ (t_0 corresponds to 12 am) is $x(t) = \alpha \ln(t/t_0)$. Then the time at which this snowplow was at x was

$$t_1(x) = t_0 e^{x/\alpha}.$$

   After the first snowplow has passed through x (but before the second snowplow arrives at x), the snow height at a later time t is proportional to $t - t_1(x)$. Thus the second snowplow has a velocity $dx/dt = \alpha/[t - t_1(x)]$, and an initial condition $x(t_0 + 1) = 0$. That ODE can be written as $dt/dx = (t - t_0 e^{x/\alpha})/\alpha$ (with initial condition $t(0) = t_0 + 1$), which gives $d(te^{-x/\alpha})/dx = -t_0/\alpha$. Then the solution of the corresponding IVP is

$$t = t_2(x) = \left(t_0 + 1 - \frac{t_0 x}{\alpha}\right) e^{x/\alpha}.$$

   This is the time $t_2(x)$ at which the second snowplow goes through x. The third snowplow has a velocity $dx/dt = \alpha/[t - t_2(x)] = \alpha/[t - (t_0 + 1 - t_0 x/\alpha)e^{x/\alpha}]$ and it satisfies $x(t_0 + 2) = 0$. As before, we rewrite the ODE as $dt/dx = [t - (t_0 + 1 - t_0 x/\alpha)e^{x/\alpha}]/\alpha$, which gives $d(te^{-x/\alpha})/dx = -(t_0 + 1)/\alpha + t_0 x/\alpha^2$. Taking into account that $t(0) = t_0 + 2$ for the third snowplow, the solution of the IVP is

$$t = t_3(x) = \left[t_0 + 2 - (1 + t_0)\frac{x}{\alpha} + \frac{t_0 x^2}{2\alpha^2}\right] e^{x/\alpha}.$$

At time $t = t_c > t_0 + 2$ all the snowplows are at the same point x, so that $t_1(x) = t_2(x) = t_3(x)$. $t_1(x) = t_2(x)$ yields $t_0 = t_0 + 1 - t_0 x/\alpha$, which implies $x = \alpha/t_0$. $t_2(x) = t_3(x)$ yields $t_0 + 1 - t_0 x/\alpha = t_0 + 2 - (1 + t_0)x/\alpha + t_0 x^2/(2\alpha^2)$ and therefore $0 = 1 - x/\alpha + t_0 x^2/(2\alpha^2)$, thereby producing $0 = 1 - t_0^{-1} + 1/(2t_0)$. Then $t_0 = 1/2$ and it started snowing at 11:30 a.m.

2. Use Taylor expansions and the mean value theorem to calculate the truncation error of the modified Euler method (mid-point method), consisting of replacing the IVP

$$\frac{du}{dt} = f(u,t), \quad u(t_0) = u_0,$$

by the scheme:

$$w_{j+1} = w_j + h\, f\left(w_j + \frac{h}{2} f(w_j, t_j), t_j + \frac{h}{2}\right), \quad w_0 = u_0,$$

with $t_{j+1} = t_j + h$ ($j = 0, 1, \ldots, N-1$; $h = \frac{T - t_0}{N}$). What is the order of the method?

Solution. By definition, the truncation error of the mid-point scheme is

$$\tau_{j+1}(h) = \frac{u(t_j + h) - u(t_j)}{h} - f\left(u(t_j) + \frac{h}{2} f(t_j, u(t_j)), t_j + \frac{h}{2}\right).$$

We now use the Taylor theorem to expand $\tau_{j+1}(h)$ to order h^2:

$$u(t_j + h) = u(t_j) + h u'(t_j) + \frac{h^2}{2} u''(t_j) + \frac{h^3}{6} u'''(\xi_j),$$

$$f\left(u(t_j) + \frac{h}{2} f(t_j, u(t_j)), t_j + \frac{h}{2}\right) = f(u(t_j), t_j) + h \frac{d}{dh}\, f\left(u(t_j) + \frac{h}{2} f(t_j, u(t_j)), t_j + \frac{h}{2}\right)\Big|_{h=0} + \frac{h^2}{2}\, \frac{d^2}{dh^2}\, f\left(u(t_j) + \frac{h}{2} f(t_j, u(t_j)), t_j + \frac{h}{2}\right)\Big|_{h=2\eta}.$$

Using the chain rule, we find

$$u''(t) = \frac{d}{dt} f(u(t), t) = f_u(u(t), t) u'(t) + f_t(u(t), t) = f_u(u(t), t) f(u(t), t) + f_t(u(t), t),$$

and

$$\frac{d}{dh}\, f\left(u(t_j) + \frac{h}{2} f(t_j, u(t_j)), t_j + \frac{h}{2}\right)\Big|_{h=0} = \frac{1}{2} f_u(u(t), t) f(u(t), t) + \frac{1}{2} f_t(u(t), t),$$

with similar formulas for the second derivative. Insertion in the truncation error produces

$$\tau_{j+1}(h) = u'(t_j) + \frac{h u''(t_j)}{2} + \frac{h^2}{6} u'''(\xi_j) - f(u_j, t_j) - \frac{h}{2}[f_u(u_j, t_j) f(u_j, t_j) + f_t(u_j, t_j)] + \frac{h^2}{8}[f_{uu}(u_j + \eta f(u_j, t_j), t_j + \eta)(f(u_j, t_j))^2 + 2 f_{ut}(u_j + \eta f(u_j, t_j), t_j + \eta) f(u_j, t_j) + f_{tt}(u_j + \eta f(u_j, t_j), t_j + \eta)],$$

where we have used the abbreviation $u_j = u(t_j)$, and, according to the generalized mean value theorem, ξ_j and η are some numbers in the intervals $t_j \le \xi_j \le t_j + h$, and $0 \le \eta \le h/2$, respectively. Using $u'(t_j) = f(u_j, t_j)$ and the previous formula, $u''(t_j) = f_u(u_j, t_j)f(u_j, t_j) + f_t(u_j, t_j)$, the terms of order 1 and h cancel and the leftover terms in the truncation error have the form h^2Q, where Q contains first and second partial derivatives of f. Assuming that all second partial derivatives exist and are continuous, Q is a continuous function of t in the interval $t_0 \le t \le t_{max}$, and therefore $|\tau_{j+1}(h)| \le Ch^2$, where $C = max_{[t_0,t_{max}]}|Q|$. Thus the modified Euler scheme is a second-order method.

3. Plot the phase portrait and classify the fixed point of the following linear system:

$$\begin{cases} \frac{dx}{dt} = -3x + 4y, \\ \frac{dy}{dt} = -2x + 3y. \end{cases}$$

If the eigenvectors are real, indicate them in your sketch.

Solution. The coefficient matrix has zero trace and determinant -1. Its eigenvalues are ± 1 and the corresponding eigenvectors are

$$\underline{\Phi}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \underline{\Phi}_{-1} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

The origin is a saddle node and its separatrices have slopes 1 (unstable separatrix) and 1/2 (stable separatrix). With these data, it is immediate to draw the phase portrait.

4. (a) By reduction of order, find the general solution of $t^2u'' - 4tu' + 6u = t^4 \sin t$ after observing that $u_1 = t^2$ is a solution of the associated homogeneous equation. (b) Use the result to solve the previous ODE with boundary conditions $u(-\pi) = 0$, $u(\pi) = 2$.

Solution. (a) We set $u(t) = t^2v(t)$ and substitute in the ODE thereby getting $t^4v'' + 4t^3v' - 4t^3v' = t^4 \sin t$, i.e., $v'' = \sin t$ that produces (after integrating twice): $v(t) = -\sin t + at + b$, where a and b are arbitrary numbers. Then the general solution of the ODE is $u(t) = -t^2 \sin t + at^3 + bt^2$. (b) $0 = u(-\pi) = \pi^2(-a\pi + b)$, $2 = u(\pi) = \pi^2(a\pi + b)$. Then $a = 1/\pi^3$, $b = 1/\pi^2$. The solution of the BVP is $u(t) = -t^2 \sin t + (t/\pi)^3 + (t/\pi)^2$.

5. Find the general solution of the ODE

$$\frac{d^2u}{dt^2} + \frac{du}{dt} - 2u = \cos^2 t.$$

Reminder: $\cos^2 t = \frac{1+\cos(2t)}{2}$.

Solution. Since $\cos^2 t = (1+\cos 2t)/2$, the method of undetermined coefficients would give $u_p = A + B\cos 2t + C \sin 2t$ which, inserted in the ODE produces $u_p'' + u_p' - 2u_p =$

$(2C-6B)\cos 2t-(6C+2B)\sin 2t-2A = 1/2(1+\cos 2t)$. *Then* $A = -1/4$, $B = -3C$ *and* $2C - 6B = 1/2$, *so that* $C = 1/40$, $B = -3/40$*:*

$$u_p(t) = -\frac{1}{4} - \frac{3}{40}\cos 2t + \frac{1}{40}\sin 2t.$$

The solution of the homogeneous ODE $u_h = e^{\lambda t}$ *gives* $\lambda^2 + \lambda - 2 = 0$, *and therefore* $\lambda = 1, -2$. *Then the general solution of the ODE is*

$$u(t) = -\frac{1}{4} - \frac{3}{40}\cos 2t + \frac{1}{40}\sin 2t + c_1 e^t + c_2 e^{-2t}.$$

A faster method consists of solving the ODE $u'' + u' - 2u = 1/2 + e^{i2t}/2$ *assuming* $U = A + De^{i2t}$ *and then taking the real part, so that* $u_p(t) = Re\, U(t)$. *We again find* $A = -1/4$ *and* $(-4 + 2i - 2)D = 1/2$, *so that* $D = \frac{1}{4(i-3)}$. *Then we obtain* $u_p = -1/4 + \frac{1}{4}Re(-3 - i)e^{i2t}/10 = \frac{1}{40}[\sin(2t) - 3\cos(2t)] - 1/4$, *which is the same particular solution as before.*
Variation of parameters is more laborious. Two independent solutions of the homogeneous problem are $u_1 = e^t$ *and* $u_2 = e^{-2t}$ *with Wronskian* $W(t) = -3e^{-t}$. *Variation of parameters with* $u = v_1 e^t + v_2 e^{-2t}$ *gives* $v_1' e^t + v_2' e^{-2t} = 0$, *so that* $u' = e^t v_1 - 2e^{-2t} v_2$. *One more differential produces* $u'' = e^t(v_1' + v_1) - 2e^{-2t}(v_2' - 2v_2) = 3e^t v_1' + e^t v_1 + 4e^{-2t} v_2$. *Substituting into the ODE, we get* $\cos^2 t = (1 + \cos 2t)/2 = 3e^t v_1'$, *from which* $v_1' = e^{-t}(1 + \cos 2t)/6$. *Then*

$$v_1 = -\frac{e^{-t}}{6}\left(1 + \frac{1}{5}\cos 2t - \frac{2}{5}\sin 2t\right) + c_1,$$

where we have used the integral $J(-1)$, *defined through*

$$J(a) = \int e^{at}\cos(2t)dt = Re\int e^{(a+2i)t}dt = Re\frac{e^{(a+2i)t}}{a+2i} = e^{at}\frac{a\cos(2t) + 2\sin(2t)}{a^2+4}.$$

Now $v_2' = -v_1' e^{3t} = -e^{2t}(1 + \cos 2t)/6$, *which integrated using* $J(2)$ *produces*

$$v_2 = -\frac{e^{2t}}{24}(2 + \cos 2t + \sin 2t) + c_2.$$

Then, after some algebra, $u(t) = v_1(t)e^t + v_2(t)e^{-2t}$ *yields the same solution as before.*

AEROSPACE ENGINEERING.
Advanced Mathematics. Final exam. January 16th, 2012.

1. Solve the inhomogeneous heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + e^{-t}, \quad 0 < x < \pi,$$

with boundary and initial conditions $u(0,t) = 0$, $u(\pi,t) = 1$, $u(x,0) = 0$.

Solution. Let us split the solution as $u = \frac{x}{\pi} + v(x,t)$. *$v$ satisfies the same heat equation with homogeneous boundary conditions* $v(0,t) = 0 = v(\pi,t)$ *and initial condition* $v(x,0) = -\frac{x}{\pi}$. *Inserting the eigenfunction expansion,* $v(x,t) = \sum_{n=1}^{\infty} b_n(t) \sin nx$, *into the heat equation for v, we obtain*

$$\frac{db_n}{dt} + n^2 b_n = \frac{2}{\pi}\int_0^{\pi} e^{-t} \sin nx\, dx = \frac{2}{n\pi} e^{-t}[1-(-1)^n].$$

This is equivalent to

$$\frac{d}{dt}\left(e^{n^2 t} b_n\right) = \frac{2}{n\pi} e^{(n^2-1)t}[1-(-1)^n] \Longrightarrow b_n(t) = k_n e^{-n^2 t} + \frac{4e^{-t}}{\pi} \begin{cases} \frac{1}{n(n^2-1)}, & \text{odd } n \neq 1, \\ t, & n = 1, \\ 0, & \text{even } n. \end{cases}$$

Now, $b_n(0) = -\frac{2}{\pi}\int_0^{\pi} \frac{x}{\pi} \sin nx\, dx = (-1)^n \frac{2}{\pi n}$, *which yields*

$$k_n = (-1)^n \frac{2}{\pi n} - \frac{4}{\pi} \begin{cases} \frac{1}{n(n^2-1)}, & \textit{odd } n \neq 1, \\ 0, & \textit{else.} \end{cases}$$

We thus find

$$u(x,t) = \frac{x}{\pi} + \sum_{n=1}^{\infty} (-1)^n \frac{2e^{-n^2 t}}{\pi n} \sin nx + \frac{4te^{-t}}{\pi} \sin x + \frac{1}{\pi} \sum_{m=1}^{\infty} \frac{e^{-t} - e^{-(2m+1)^2 t}}{(2m+1)m(m+1)} \sin[(2m+1)x].$$

2. Consider the first-order wave equation $\partial u/\partial t + c\partial u/\partial x = 0$.

   (a) Discretize this equation by using a centered difference in time and a centered difference in space. What is the truncation error of the resulting scheme?

   (b) Analyze the stability of the resulting scheme (without boundary conditions).

Solution. (a) $\frac{u_j^{(m+1)}-u_j^{(m-1)}}{2\Delta t}+c\frac{u_{j+1}^{(m)}-u_{j-1}^{(m)}}{2\Delta x}=0$ *gives* $u(m+1)_j = u_j^{(m-1)} - s(u_{j+1}^{(m)} - u_{j-1}^{(m)})$ *with* $s = c\Delta t/\Delta x$. *Inserting* $u(m)_j = u(j\Delta x, m\Delta t)$ *in the previous scheme and Taylor expanding,*

$$\frac{u(j\Delta x,(m+1)\Delta t)-u(j\Delta x,(m-1)\Delta t)}{2\Delta t}+c\frac{u((j+1)\Delta x,m\Delta t)-u((j-1)\Delta x,m\Delta t)}{2\Delta x}$$
$$=\frac{(\Delta t)^2}{6}\frac{\partial^3 u}{\partial t^3}(j\Delta x,(m+\tau)\Delta t)+c\frac{(\Delta x)^2}{6}\frac{\partial^3 u}{\partial x^3}(\Delta x(j+\xi),m\Delta t)$$

(τ and ξ are between -1 and 1), and therefore the truncation error is quadratic, comprising a term of order $(\Delta t)^2$ plus a term of order $(\Delta x)^2$.
(b) Substituting $u_j^{(m)} = Q^m e^{ijk}$ *in the difference equation, we find* $Q = Q^{-1} - s(e^{ik} - e^{-ik}) = Q^{-1} - 2is\sin k$. *This is the second order equation* $Q^2 + 2isQ\sin k - 1 = 0$ *whose solutions are:* $Q_\pm = -is\sin k \pm \sqrt{1-s^2\sin^2 k}$. *For* $s|\sin k| < 1$, *we get complex* $Q_\pm$ *such that* $|Q_\pm|^2 = 1$. *For* $s|\sin k| = 1$, *we also get* $|Q_\pm| = 1$, *whereas for* $s|\sin k| > 1$, *the square root is purely imaginary and* $|Q_-| > 1$. *The two first conditions hold if* $s \le 1$ *and therefore the scheme is stable if* $c \le \frac{\Delta x}{\Delta t}$.

3. A semi-infinite homogeneous string has a point mass M attached to its left end at $x = 0$. The string linear mass density and tension are ρ and T, respectively. Then the wave speed is $c = \sqrt{T/\rho}$. The mass is sustained by a massless spring with stiffness constant k. Then the vertical displacement of the mass M is equal to the vertical displacement of the string at $x = 0$: $y(t) = u(0, t)$. Consider that the string and the mass are initially at rest but that the mass has an initial vertical velocity v_0. The initial velocity profile of the string is zero for $x > 0$.

   (a) Write the equation of motion for $y(t)$ by using Newton's second law considering the force $T\frac{\partial u}{\partial x}(0, t)$ that the string exerts on the mass M.

   (b) In the previous equation, calculate the term that depends on u at $x = 0$ in terms of $y(t)$ by solving the wave equation for $x > 0$ with boundary condition $u(0, t) = y(t)$. Substituting this term in the equation of motion for the mass, obtain an ordinary differential equation for $y(t)$.

   (c) Solve the initial-value problem for $y(t)$ in the cases $\rho T > 4Mk$ and $\rho T < 4Mk$.

Solution. (a) The string exerts a force $T\frac{\partial u}{\partial x}$ *at* $x = 0$ *over the mass and therefore Newton's second law yields*

$$M\ddot{y} = -ky + T\frac{\partial u}{\partial x}(0, t),$$

to be solved with initial conditions $y(0) = 0$ *and* $\dot{y}(0) = v_0$. *Note that we are not considering gravity. A constant term* $-Mg$ *in the right-hand side of the previous expression can be absorbed by redefining* $z = y - Mg/k$, $v = u - Mg/k$, *so that* $z(t)$ *and* $v(x, t)$ *obey the same equations, initial and boundary conditions as our functions* $y(t)$ *and* $u(x, t)$, *respectively.*

(b) To find $\frac{\partial u}{\partial x}(0,t)$, we need to solve the following IBVP for the wave equation:

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, & x > 0,\, t > 0, \\ u(0,t) = y(t), & x = 0, \\ u(x,0) = 0, & \frac{\partial u}{\partial t}(x,0) = 0. \end{cases}$$

For $x > ct$, the trivial initial conditions give the solution $u = 0$. For $x < ct$, the boundary condition $u(0,t) = y(t)$ produces the solution $u(x,t) = y(t - x/c)$. Then $\frac{\partial u}{\partial x} = -(1/c)\dot{y}(t - x/c)$ and $T\frac{\partial u}{\partial x}(0,t) = -\frac{T}{c}\dot{y}$. Inserting this in the equation for $y(t)$, we find

$$M\ddot{y} = -ky - \frac{T}{c}\dot{y} \Longrightarrow \ddot{y} + \frac{T}{Mc}\dot{y} + \frac{k}{M}y = 0.$$

Note that the coupling with the string has yielded a friction force on the mass: the mass motion produces waves that escape along the string thereby taking away energy.
(c) The general solution of the previous ODE is found by inserting $y = e^{\lambda t}$, which gives $\lambda^2 + \frac{T}{Mc}\lambda + \frac{k}{M} = 0$, and therefore

$$\lambda = -\frac{T}{2Mc} \pm \sqrt{\frac{T^2}{4M^2c^2} - \frac{k}{M}} = -\frac{T}{2Mc} \pm \frac{\sqrt{\rho T - 4kM}}{2M},$$

where we have used $T = \rho c^2$.
For $\rho T < 4kM$, the square root is purely imaginary and the general solution of the ODE is

$$y(t) = e^{-Tt/(2Mc)}\left(a\cos\frac{\sqrt{\rho T - 4kM}}{2M}t + b\sin\frac{\sqrt{\rho T - 4kM}}{2M}t\right).$$

The boundary condition $y(0) = 0$ gives $a = 0$ and $\dot{y}(0) = v_0$ produces $v_0 = b\frac{\sqrt{\rho T - 4kM}}{2M}$. Thus for $\rho T < 4kM$,

$$y(t) = \frac{2Mv_0}{\sqrt{\rho T - 4kM}}e^{-Tt/(2Mc)}\sin\frac{\sqrt{\rho T - 4kM}}{2M}t.$$

Similarly, for $\rho T > 4kM$, the general solution of the ODE is

$$y(t) = e^{-Tt/(2Mc)}\left(a\cosh\frac{\sqrt{4kM - \rho T}}{2M}t + b\sinh\frac{\sqrt{4kM - \rho T}}{2M}t\right).$$

The boundary condition $y(0) = 0$ gives $a = 0$ and $\dot{y}(0) = v_0$ produces $v_0 = b\frac{\sqrt{4kM - \rho T}}{2M}$. Thus for $\rho T > 4kM$,

$$y(t) = \frac{2Mv_0}{\sqrt{4kM - \rho T}}e^{-Tt/(2Mc)}\sinh\frac{\sqrt{4kM - \rho T}}{2M}t.$$

4. Solve the Poisson equation $\nabla^2 u = y\cos x$ in $(0,\pi)\times(0,L)$ with boundary conditions

$$\frac{\partial u}{\partial x}(0,y) = 0, \quad u(x,0) = 0,$$
$$\frac{\partial u}{\partial x}(\pi,y) = 0, \quad u(x,L) = f(x).$$

Solution. We use the eigenfunction expansion $u(x,y) = \frac{a_0(y)}{2} + \sum_{n=1}^{\infty} a_n(y)\cos nx$ *which is appropriate for the homogeneous Neumann conditions at* $x = 0,\, \pi$*. Inserting in the Poisson equation and differentiating term by term, we obtain*

$$a_n'' - n^2 a_n = y\delta_{n1},$$

where $\delta_{n1} = 1$ *if* $n = 1$ *and 0 otherwise. The boundary conditions for this ODE are* $a_n(0) = 0$ *and* $a_n(L) = \frac{2}{\pi}\int_0^\pi f(s)\cos ns\, ds$*.*
For $n = 0$*, we get* $a_0(y) = \alpha_0 y$ *with* $\alpha_0 = \frac{2}{L\pi}\int_0^\pi f(s)\, ds$*. For* $n > 1$*, the solution satisfying* $a_n(0) = 0$ *is* $a_n(y) = \beta_n \sinh ny$*. The other boundary condition produces* $\beta_n = \frac{2}{\pi \sinh nL}\int_0^\pi f(s)\cos ns\, ds$*. Lastly, for* $n = 1$*, a particular solution of* $a_1'' - a_1 = y$ *is* $-y$*, so that* $a_1(y) = -y + \alpha_1 \cosh y + \beta_1 \sinh y$*. The boundary conditions yield* $\alpha_1 = 0$ *and* $-L + \beta_1 \sinh L = \frac{2}{\pi}\int_0^\pi f(s)\cos s\, ds$*. Then*

$$\beta_1 = \frac{1}{\sinh L}\left(L + \frac{2}{\pi}\int_0^\pi f(s)\cos s\, ds\right).$$

Putting together these results, we obtain the solution

$$u(x,y) = \frac{y}{\pi L}\int_0^\pi f(s)\, ds + \left[\frac{\sinh y}{\sinh L}\left(L + \frac{2}{\pi}\int_0^\pi f(s)\cos s\, ds\right) - y\right]\cos x + \sum_{n=2}^{\infty}\frac{2}{\pi \sinh nL}\left(\int_0^\pi f(s)\cos ns\, ds\right)\sinh ny\cos nx.$$

The same solution is obtained if we separate $u(x,y) = -y\cos x + v(x,t)$*, check that* $v(x,y)$ *solves the Laplace equation with the same boundary conditions as* u *does and then use the method of 1D eigenfunction expansions to get* $v(x,y)$*.*

Another method. Multiply the Poisson equation by the eigenfunction $\cos nx \sin\frac{m\pi y}{L}$ *and integrate over* x *and* y*. The right hand side gives:*

$$\int_0^\pi \cos nx\cos x dx\int_0^L y\sin\frac{m\pi y}{L}dy = \frac{\pi}{2}\delta_{n1}(-1)^{m+1}\frac{L^2}{m\pi}.$$

The integral over the left hand side is done by parts:

$$\begin{aligned}
\int_0^\pi\int_0^L (u_{xx}+u_{yy})\cos nx\sin\frac{m\pi y}{L}dxdy &= \int_0^L [u_x\cos nx]|_0^\pi \sin\frac{m\pi y}{L}dy \\
&+ \int_0^\pi [u_y \sin\frac{m\pi y}{L}]|_0^L \cos nx dx + n\int_0^\pi\int_0^L u_x \sin nx\sin\frac{m\pi y}{L}dxdy \\
-\frac{m\pi}{L}\int_0^\pi\int_0^L u_y\cos nx\cos\frac{m\pi y}{L}dxdy &= n\int_0^L [u\sin nx]|_0^\pi \sin\frac{m\pi y}{L}dy \\
-\frac{m\pi}{L}\int_0^\pi [u\cos\frac{m\pi y}{L}]|_0^L\cos nx dx - \left(n^2+\frac{m^2\pi^2}{L^2}\right)\int_0^\pi\int_0^L u\cos nx\sin\frac{m\pi y}{L}dxdy & \\
= -\frac{m\pi}{L}(-1)^m\int_0^\pi f(x)\cos nx dx - \left(n^2+\frac{m^2\pi^2}{L^2}\right)\int_0^\pi\int_0^L u\cos nx\sin\frac{m\pi y}{L}dxdy. &
\end{aligned}$$

Here we have denoted $u_x = \partial u/\partial x$, $u_{xx} = \partial^2 u/\partial x^2$, etc. We have found

$$\left(n^2 + \frac{m^2\pi^2}{L^2}\right)\int_0^\pi \int_0^L u \cos nx \sin \frac{m\pi y}{L} dxdy = (-1)^m \left[\frac{L^2}{2m}\delta_{n1} - \frac{m\pi}{L}\int_0^\pi f(x)\cos nx dx\right].$$

This result yields the following eigenfunction expansion of the solution:

$$u(x,y) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} c_{nm}\cos nx \sin\frac{m\pi y}{L},$$
$$c_{n,m} = \frac{4}{\pi L(1+\delta_{n0})}\int_0^\pi\int_0^L u\cos nx\sin\frac{m\pi y}{L}dxdy,$$
$$c_{0m} = \frac{2(-1)^m}{m\pi^2}\int_0^\pi f(x)dx, \quad c_{nm} = \frac{4(-1)^m}{n^2+\frac{m^2\pi^2}{L^2}}\left[\frac{L}{2m\pi}\delta_{n1} - \frac{m}{L^2}\int_0^\pi f(x)\cos nx\,dx\right] (n\neq 0).$$

To check that both results are the same, substitute the Fourier sine series

$$y = -\sum_{m=1}^{\infty}\frac{2L(-1)^m}{m\pi}\sin\frac{m\pi y}{L}, \quad \sinh ny = -\sum_{m=1}^{\infty}\frac{2m\pi(-1)^m}{L^2}\frac{\sinh nL}{n^2+\frac{m^2\pi^2}{L^2}}\sin\frac{m\pi y}{L},$$

in the formula for $u(x,y)$ obtained with the method of 1D eigenfunction expansions.

5. Find the general solution of the inhomogeneous system of ODEs:

$$\dot{x} = 4x - 3y,$$
$$\dot{y} = 8x - 6y - 1.$$

Solution. The trace and determinant of the coefficient matrix are -2 and 0, respectively. Thus their eigenvalues are -2 and 0 and the general solution of the homogeneous system is $\underline{x}(t) = \underline{a} + \underline{b}e^{-2t}$, where $\underline{a}$ and $\underline{b}$ are arbitrary vectors. We can use now the method of variation of parameters:

$$\underline{x}(t) = a(t)\begin{pmatrix}\frac{3}{4}\\ 1\end{pmatrix} + b(t)e^{-2t}\begin{pmatrix}1\\2\end{pmatrix}.$$

Inserting this in the system, we obtain

$$\dot{a}\begin{pmatrix}\frac{3}{4}\\1\end{pmatrix} + \dot{b}e^{-2t}\begin{pmatrix}1\\2\end{pmatrix} = -\begin{pmatrix}0\\1\end{pmatrix}.$$

Substituting $\dot{a} = -\frac{4}{3}e^{-2t}\dot{b}$ in the second equation, we get $(2-4/3)e^{-2t}\dot{b} = -1$, i.e., $\dot{b} = -\frac{3}{2}e^{2t}$. The solution of this ODE is

$$b(t) = \beta - \frac{3}{4}e^{2t}.$$

Then $\dot{a} = -\frac{4}{3}e^{-2t}\dot{b} = 2$, and

$$a(t) = \alpha + 2t.$$

We have found

$$\underline{x}(t) = \begin{pmatrix} \frac{3}{4}\alpha + \frac{3}{2}t \\ \alpha + 2t \end{pmatrix} + \Big(\beta e^{-2t} - \frac{3}{4}\Big)\begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

We can also use the method of undetermined coefficients. Since an appropriate constant vector (the eigenvector corresponding to the eigenvalue 0) is a solution of the homogeneous system, the method of undetermined coefficients suggests the following form for the solution of the inhomogeneous system:

$$\underline{x}(t) = \underline{a} + \underline{b}e^{-2t} + \underline{c}t.$$

Inserting this in the system

$$\Big(\underline{\underline{I}}\frac{d}{dt} - \underline{\underline{A}}\Big)\underline{x} = -\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \underline{\underline{A}} = \begin{pmatrix} 4 & -3 \\ 8 & -6 \end{pmatrix},$$

we find

$$-\underline{\underline{A}}\underline{a} - e^{-2t}\Big(2\underline{\underline{I}} + \underline{\underline{A}}\Big)\underline{b} + \underline{c} - t\underline{\underline{A}}\underline{c} = -\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

From this equation, we get

$$\Big(2\underline{\underline{I}} + \underline{\underline{A}}\Big)\underline{b} = 0, \tag{1}$$

$$\underline{\underline{A}}\underline{c} = 0, \tag{2}$$

$$\underline{\underline{A}}\underline{a} = \underline{c} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{3}$$

Calculating the eigenvectors of $\underline{\underline{A}}$ corresponding to the eigenvalues -2 and 0, we obtain from (1) and (2):

$$\underline{b} = \begin{pmatrix} b_1 \\ 2b_1 \end{pmatrix}, \quad \underline{c} = \begin{pmatrix} \frac{3}{4}c_2 \\ c_2 \end{pmatrix},$$

respectively. (3) can be written as

$$\begin{aligned} 4a_1 - 3a_2 &= \tfrac{3}{4}c_2, \\ 8a_1 - 6a_2 &= c_2 + 1. \end{aligned}$$

We now multiply the first of these equations by 2 and subtract the second one from it, thereby obtaining $3c_2/2 - c_2 - 1 = 0$. Thus $c_2 = 2$. The first equation then yields $a_1 = \frac{3}{4}a_2 + \frac{3}{8}$. Putting everything together, we have found the general solution:

$$\underline{x}(t) = \begin{pmatrix} \frac{3}{4}a_2 + \frac{3}{8} \\ a_2 \end{pmatrix} + 2t\begin{pmatrix} \frac{3}{4} \\ 1 \end{pmatrix} + b_1 e^{-2t}\begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

in which a_2 and b_1 are arbitrary constants. This is the same result as given by the other method if we identify $a_2 = \alpha - 3/2$, $b_1 = \beta$.

Yet another method. Note that

$$2\dot{x} - \dot{y} = 1 \Longrightarrow 2x - y = t + c.$$

Then $y = 2x - t - c$ *which, inserted in the equation for* x*, yields*

$$\dot{x} = 4x - 6x + 3t + 3c = -2x + 3(t + c).$$

A particular solution of this ODE is $x_p = \frac{3t}{2} - \frac{3}{4} + \frac{3c}{2}$*. Then*

$$x(t) = \frac{3t}{2} - \frac{3}{4} + \frac{3c}{2} + de^{-2t},$$

and

$$y(t) = 2x - t - c = 3t - \frac{3}{2} + 3c + 2de^{-2t} - t - c = 2t - \frac{3}{2} + 2c + 2de^{-2t}.$$

This solution of the system coincides with that given by the the method of variation of parameters if $c = \alpha/2$ *and* $d = \beta$*.*

AEROSPACE ENGINEERING.
Advanced Mathematics. Midterm exam. October 25th, 2012.

Select and solve four out of the following five problems.

1. A spherical rain drop falls in the gravitational field starting from rest. Provided the drop picks up water vapor at a rate proportional to its surface and has zero initial radius, prove that the drop falls with constant acceleration and find it. Hint: Use that the force is the time derivative of the linear momentum (mass times velocity).

   Solution. We have $d(mv)/dt = -mg$ *and therefore*

   $$\frac{dv}{dt} = -g - v\frac{d}{dt}\ln m = -g - 3v\frac{d}{dt}\ln r,$$

   because $m = \frac{4}{3}\rho\pi r^3$. *We also have*

   $$\frac{dm}{dt} = k(4\pi r^2) \Longrightarrow \frac{dr}{dt} = \frac{k}{\rho}.$$

   This gives $r(t) = \frac{k}{\rho}t$ *because* $r(0) = 0$. *Then*

   $$\frac{dv}{dt} = -g - \frac{3v}{t}.$$

   Multiplying by the integrating factor t^3, *we obtain*

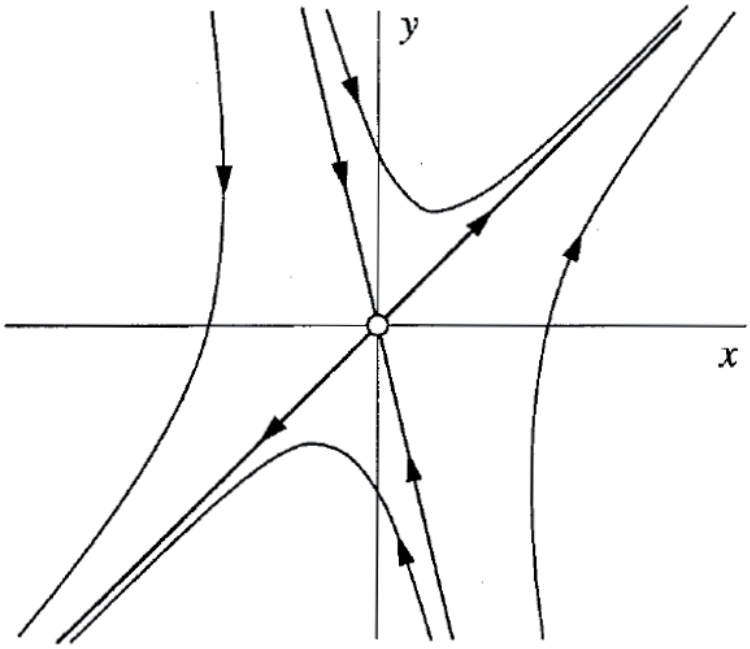
   $$t^3\frac{dv}{dt} + 3t^2v = \frac{d}{dt}(t^3v) = -gt^3.$$

   Integrating and using the initial condition, we get $t^3v = -gt^4/4$, *from* $v(t) = -gt/4$. *Then*

   $$\frac{dv}{dt} = -\frac{g}{4},$$

   which proves that the drop falls with constant acceleration $g/4$.

2. Find all functions $u(t)$, such that $tu'' - (1+t)u' + u = 0$. Hint. Try reduction of order if you can guess one solution.



Solution. By inspection, two solutions are $u_1 = 1 + t$ *and* $u_2 = e^t$. *They are independent and therefore the general solution is* $u(t) = c_1(1+t) + c_2e^t$. *If you can guess only one of them, for example,* u_2, *try* $u = u_2v$. *This yields* $tv'' + (t-1)v' = 0$, *i.e.,* $v''/v' = (1-t)/t$ *that gives* $\ln v' = \ln t - t + \ln c_2$. *Then* $v' = c_2te^{-t}$ *and* $v = -c_2(t+1)e^{-t} + c_1$, *and therefore* $u = c_1e^t - c_2(t+1)$ *as before.*

3. Plot the phase portrait and classify the fixed point of the following linear system:

$$\begin{cases} \frac{dx}{dt} = x + y, \\ \frac{dy}{dt} = 4x - 2y. \end{cases}$$

If the eigenvectors are real, indicate them in your sketch.

Solution. The characteristic equation is $\lambda^2 + \lambda - 6 = 0$ *and the eigenvalues are 2, with eigenvector* $(1, 1)$, *and -3, with eigenvector* $(1, -4)$. *The origin is a saddle point and the phase portrait can be drawn depicting the separatrices which are straight lines with directions given by the eigenvectors. See the figure where the arrows indicate the direction of increasing* t.

4. Find the general solution of the system of ODEs

$$\begin{cases} \frac{dx}{dt} = x + y + e^t, \\ \frac{dy}{dt} = 4x - 2y + 1. \end{cases}$$

Solution. We know that the eigenvalues are 2, with eigenvector $(1, 1)$, *and -3, with eigenvector* $(1, -4)$. *The fundamental matrix is therefore*

$$\underline{\underline{\Psi}} = \begin{pmatrix} e^{-3t} & e^{2t} \\ -4e^{-3t} & e^{2t} \end{pmatrix}$$

Its inverse is

$$\underline{\underline{\Psi}}^{-1} = \begin{pmatrix} e^{-3t} & e^{2t} \\ -4e^{-3t} & e^{2t} \end{pmatrix}$$

Then

$$\int \underline{\underline{\Psi}}^{-1}\underline{F}(t)dt = \frac{1}{5}\int \begin{pmatrix} e^{3t} & -e^{3t} \\ 4e^{-2t} & e^{-2t} \end{pmatrix}\begin{pmatrix} e^{t} \\ 1 \end{pmatrix} dt = \frac{1}{5}\int \begin{pmatrix} e^{4t}-e^{3t} \\ 4e^{-t}+e^{-2t} \end{pmatrix} dt.$$

Using now $\underline{x}(t) = \underline{\underline{\Psi}}(t)\int \underline{\underline{\Psi}}^{-1}\underline{F}(t)dt$, *we find a particular solution to which we add the general solution of the homogeneous system:*

$$\underline{x}(t) = \begin{pmatrix} -\frac{3}{4}e^{t} - \frac{1}{6} \\ -e^{t} + \frac{1}{6} \end{pmatrix} + \underline{\underline{\Psi}}\,\underline{c}.$$

The method of undetermined coefficients would give $\underline{x}_p = \underline{a}e^t + \underline{b}$ *which, inserted in the system produces* $(\underline{a} - \underline{\underline{A}}\underline{a} = (1,0)$ *and* $-\underline{\underline{A}}\underline{b} = (0,1)$. *Then* $\underline{a} = (\underline{\underline{I}} - \underline{\underline{A}})^{-1}(1,0) = ((-\frac{3}{4}, -1)$, $\underline{b} = -\underline{\underline{A}}^{-1}(0,1) = \frac{1}{6}(-1,1)$, *and this yields the same particular solution as before.*

5. Find the general solution of the ODE: $u'' + 4u' + 3u = \sin t$. Then determine the arbitrary constants so that $u(0) = 0$ and $u'(0) = 0$.

Solution. The homogeneous ODE has solution $u_h = c_1e^{-t} + c_2e^{-3t}$. *A particular solution of the ODE has the form:* $u_p = a\cos t + b\sin t$. *Then* $u_p'' + 4u_p' + 3u_p = (2a + 4b)\cos t + (2b - 4a)\sin t = \sin t$, *so that* $2b - 4a = 1$, $2a + 4b = 0$. *Then* $a = -1/5$, $b = 1/10$. *The general solution of the inhomogeneous ODE is*

$$u(t) = -\frac{1}{5}\cos t + \frac{1}{10}\sin t + c_1e^{-t} + c_2e^{-3t}.$$

We determine the constants c_1, c_2 *by inserting this solution in the initial conditions:*

$$c_1 + c_2 = \frac{1}{5}, \quad c_1 + 3c_2 = \frac{1}{10}.$$

$$c_1 = \frac{1}{4}, \quad c_2 = -\frac{1}{20}.$$

The solution of the IVP is

$$u(t) = -\frac{1}{5}\cos t + \frac{1}{10}\sin t + \frac{1}{4}e^{-t} - \frac{1}{20}e^{-3t}.$$

AEROSPACE ENGINEERING.
Advanced Mathematics. Final exam. January 23rd, 2013.

Select and solve four out of the following five problems.

1. A nanotube resonator may be modeled as a thin cylinder of several nanometers radius supported on two contacts. The natural transversal vibrations of a nanotube with simply supported ends satisfy the equation: $\partial_t^2 u = -a^2 \partial_x^4 u$, with boundary conditions $u(0,t) = \partial_x^2 u(0,t) = 0$, $u(L,t) = \partial_x^2 u(L,t) = 0$. Assume that the initial conditions are $u(x,0) = f(x)$ and $\partial_t u(x,0) = 0$. Solve this initial-boundary value problem specifying the natural frequencies of the nanotube. *Hint: Separate variables and use that the quartic roots of 1 are* ± 1, $\pm i$.

   Solution. Separating variables, $u(x,t) = X(x)T(t)$, *we have* $\frac{T''}{a^2 T} = -\frac{X^{(4)}}{X} = -\lambda$. *Choosing* $\lambda > 0$, *the spatial part of the solution yields the following eigenvalue problem*

$$X^{(4)} = \lambda X, \quad X(0) = X''(0) = 0, \quad X(L) = X''(L) = 0,$$

   whose solutions are $X(x) = a_1 \cosh(\lambda^{1/4}x) + a_2 \sinh(\lambda^{1/4}x) + b_1 \cos(\lambda^{1/4}x) + b_2 \sin(\lambda^{1/4}x)$. *(Trying* $X = e^{qx}$, *we get* $q^4 = \lambda$ *whose solutions are* $q/\lambda^{1/4} = \pm 1, \pm i$. *These roots provide the indicated general solution.) The boundary conditions at* $x = 0$ *produce* $a_1 + b_1 = 0$ *and* $\sqrt{\lambda}(a_1 - b_1) = 0$. *This gives* $a_1 = b_1 = 0$. *Then the boundary conditions at* $x = L$ *yield* $a_2 \sinh(\lambda^{1/4}L) + b_2 \sin(\lambda^{1/4}L) = 0$ *and* $\sqrt{\lambda}[a_2 \sinh(\lambda^{1/4}L) - b_2 \sin(\lambda^{1/4}L)] = 0$. *This gives* $a_2 = 0$ *and* $b_2 \sin(\lambda^{1/4}L) = 0$. *Then* $\lambda^{1/4}L = n\pi$, *where* $n = 1, 2, \ldots$. *Thus the eigenvalues are* $\lambda_n = (n\pi/L)^4$, $n = 1, 2, \ldots$ *with eigenfunctions* $X_n(x) = \sin \frac{n\pi x}{L}$. *If* $\lambda \leq 0$, *then* $X(x) = 0$. *The equation for* $T(t)$ *is* $T'' + T(n^2\pi^2 a/L^2)^2 = 0$ *whose general solution is* $T = A_n \cos \frac{n^2\pi^2 at}{L^2} + B_n \cos \frac{n^2\pi^2 at}{L^2}$. *The natural frequencies of the nanotube are therefore* $\omega_n = \frac{n^2\pi^2 a}{L^2}$, *with* $n = 1, 2, \ldots$. *The superposition principle yields*

$$u(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos \frac{n^2\pi^2 at}{L^2} + B_n \cos \frac{n^2\pi^2 at}{L^2} \right) \sin \frac{n\pi x}{L}.$$

   The initial condition $\partial_t u(x,0) = 0$ *gives* $B_n = 0$ *and the other initial condition produces*

$$f(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L}.$$

Then

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx,$$

and

$$u(x,t) = \sum_{n=1}^{\infty} A_n \cos\frac{n^2\pi^2 at}{L^2}\,\sin\frac{n\pi x}{L}.$$

2. The truncation error of the explicit finite difference scheme for the homogeneous heat equation $\partial_t u = k\partial_x^2 u$ is $\tau_j^{(m)}(\Delta t,\Delta x) = \frac{u_j^{(m+1)}-u_j^{(m)}}{\Delta t} - \frac{k}{(\Delta x)^2}(u_{j+1}^{(m)}+u_{j-1}^{(m)}-2u_j^{(m)})$, where $u_j^{(m)}$ should be replaced by $u(j(\Delta x),m(\Delta t))$.

   (a) Calculate $|\tau_j^{(m)}|$. For which value of $s = k(\Delta t)/(\Delta x)^2$ is this error smallest?

   (b) Calculate $|\tau_j^{(m)}|$ for the specific value of s found in (a).

   Solution. (a) Taylor expand $u_j^{(m+1)}$ and $u_{j\pm1}^{(m)}$ up to terms of order $(\Delta t)^2$ and $(\Delta x)^4$. Then use the heat equation to get rid of the time derivatives in the result:

$$\tau_j^{(m)} = \frac{\Delta t}{2}\partial_t^2 u_j^{(m)} + \frac{(\Delta t)^2}{2}\partial_t^3 u_j^{(m)} - \frac{k(\Delta x)^2}{12}\partial_x^2 u_j^{(m)} - \frac{2k(\Delta x)^4}{6!}\partial_x^6 u_j^{(m)} + \dots$$

$$\tau_j^{(m)} = \frac{k(\Delta x)^2}{2}\frac{k(\Delta x)}{2}\left(\frac{k(\Delta t)}{(\Delta x)^2}-\frac{1}{6}\right)\partial_x^4 u_j^{(m)} + \frac{k(\Delta x)^4}{6}\left(\frac{k^2(\Delta t)^2}{(\Delta x)^4}-\frac{1}{60}\right)\partial_x^6 u_j^{(m)} + \dots$$

   Clearly the error $|\tau_j^{(m)}|$ is smallest for $s = 1/6$.
   (b) Inserting $s = 1/6$ in the last formula for the error, we get $\tau_j^{(m)} = \frac{k(\Delta x)^4}{540}\partial_x^6 u_j^{(m)}$ plus higher order terms.

3. Solve

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}, \quad t>0,$$

   with initial conditions $u(x,0) = 0$, $\frac{\partial u}{\partial t}(x,0) = 0$, $c > 0$, subject to $u(x,t) = h(t)$ along $x = \frac{ct}{2}$.

   Draw the plane (x,t) with the straight lines $x = \pm ct$ and $x = ct/2$. There are four regions to consider for $t > 0$. In regions $x < -ct$ and $x > ct$ the initial conditions imply that $u(x,t) = 0$. In region $\frac{ct}{2} < x < ct$, the characteristics with positive slope passing through point (x,t) is $c(\tau - t) = \xi - x$. This line intersects $\xi = c\tau/2$ at $\xi = \frac{c\tau}{2} = ct - x$. Then the parallelogram rule implies $u(x,t) = h(2t - 2x/c)$ ($u = 0$ for the two vertices on $x = ct$). In the region $-ct < x < \frac{ct}{2}$, the characteristics with negative slope passing through point (x,t) is $c(\tau - t) = -(\xi - x)$. This line intersects $\xi = c\tau/2$ at $\xi = \frac{c\tau}{2} = \frac{x+ct}{3}$. Then the parallelogram rule implies $u(x,t) = h(\frac{2}{3c}(x+ct))$ ($u = 0$ for the two vertices on $x = -ct$).
   Another solution method. In regions $x < -ct$ and $x > ct$ the initial conditions imply that $u(x,t) = 0$. If $u(x,t) = F(x-ct) + G(x+ct)$, then $F(\xi) = G(\xi) = 0$ for $\xi > 0$ (region $x > ct$) and $F(\xi) = G(\xi) = 0$ for $\xi < 0$ (region $x < -ct$). In the region

$\frac{ct}{2} < x < ct$ *which is separated from* $x > ct$ *by the line* $x = ct$, $x - ct < 0$ *but* $x + ct > 0$ *so that* $G(x + ct) = 0$ *and* $u = F(x - ct)$ *with* $-\frac{ct}{2} < (x - ct) < 0$. *Then* $h(t) = F(\frac{ct}{2} - ct) = F(-\frac{ct}{2})$ *by the boundary condition. This implies* $F(\xi) = h(-2\xi/c)$ *for* $-\frac{ct}{2} < \xi < 0$. *Thus* $u(x,t) = h(2t - 2x/c)$ *in the region* $\frac{ct}{2} < x < ct$. *In the region* $-ct < x < \frac{ct}{2}$, $x - ct < 0$, $0 < (x + ct) < 3c/2$, *so that* $F(x - ct) = 0$ *and therefore* $u = G(x + ct)$. *At the boundary,* $h(t) = G(\frac{ct}{2} + ct) = G(\frac{3ct}{2})$. *Then* $G(\xi) = h(\frac{2\xi}{3c})$ *and* $u(x,t) = h(\frac{2(x+ct)}{3c})$.

4. The two ends of a cylindrical metal bar of length L are kept at zero temperature and its lateral surface exchanges heat with the outside (at fixed temperature u_0) according to Newton's cooling law. Initially the bar temperature is zero. Find the temperature at each point of the bar. *Hint. Use* $\partial_t u = k\partial_x^2 u - h(u - u_0)$.

   Solution. We solve the temperature equation with Dirichlet boundary conditions and zero initial condition: $u = u_0 + v(x) + w(x,t)$, *where*

$$kv'' - hv = 0, \quad v(0) = -u_0, \quad v(L) = -u_0.$$

   We get

$$v(x) = -u_0 \frac{\sinh\left[(L-x)\sqrt{\frac{h}{k}}\right] + \sinh x\sqrt{\frac{h}{k}}}{\sinh L\sqrt{\frac{h}{k}}}.$$

   Then $w(x,t)$ *satisfies the homogeneous equation with zero-data Dirichlet boundary conditions* $\partial_t w = k\partial_x^2 w - hw$, $w(0,t) = 0 = w(L,t)$, *and initial conditions* $w(x,0) = -u_0$. *Inserting the eigenfunction expansion:* $w(x,t) = \sum_{n=1}^{\infty} b_n(t) \sin\frac{n\pi x}{L}$, *we obtain the ODE* $b_n' = -(h + kn^2\pi^2/L^2)b_n$ *to be solved with initial condition* $b_n(0) = -u_0\frac{2}{L}\int_0^L \sin\frac{n\pi x}{L}dx = \frac{2u_0}{n\pi}[(-1)^n - 1]$. *The result is* $u = u_0 + v(x) + w(x,t)$, *with*

$$w(x,t) = -\frac{4hL^2u_0}{\pi k}\sum_{j=1}^{\infty} \frac{\sin\frac{(2j-1)\pi x}{L}}{(2j-1)[(2j-1)^2\pi^2 k + hL^2]} e^{-[h+(2j-1)^2\pi^2 k/L^2]t}.$$

   Another method. Eigenfunction expansion $u = \sum_{n=1}^{\infty} b_n(t)\sin\frac{n\pi x}{L}$, *the boundary and initial conditions yield*

$$b_n' + \left(k\frac{n^2\pi^2}{L^2} + h\right)b_n = \frac{2}{n\pi}[1 - (-1)^n]hu_0, \quad b_n(0) = 0.$$

   The solution is $b_n(t) = \frac{2}{n\pi}\frac{1-(-1)^n}{h+kn^2\pi^2/L^2}hu_0[1 - e^{-(h+kn^2\pi^2/L^2)t}]$, *thereby producing the equivalent form of the temperature*

$$u(x,t) = \frac{4hL^2u_0}{\pi}\sum_{j=1}^{\infty} \frac{\sin\frac{(2j-1)\pi x}{L}}{(2j-1)[hL^2 + k(2j-1)^2\pi^2]}[1 - e^{-[h+k(2j-1)^2\pi^2/L^2]t}].$$

   This can be checked by Fourier-expanding u_0 *and* $v(x)$ *above.*

5. Solve the Poisson equation:

$$\nabla^2 u = \sin(2x)\cos y,$$

with the boundary conditions

$$u(0,y) = 0, \quad u(x,0) = f(x),$$
$$u(\pi,y) = 0, \quad u(x,L) = 0.$$

Solution. The boundary conditions at $x = 0$ and π are homogeneous and therefore we can use an eigenfunction expansion $u(x,y) = \sum_{n=1}^{\infty} b_n(y)\sin(nx)$. Inserting this in the Poisson equation we find

$$\sum_{n=1}^{\infty}[b_n''(y) - n^2 b_n(y)]\sin(nx) = \sin(2x)\cos y.$$

From this equation we obtain $b_n'' - n^2 b_n = \delta_{n2}\cos y$. The boundary conditions are $b_n(L) = 0$ and $b_n(0) = f_n$, where $f_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx$. For $n \neq 2$, $b_n(y) = \beta_n \sinh[n(y-L)]$ with $\beta_n = -f_n/\sinh(nL)$. For $n = 2$, we have $b_2'' - 4b_2 = \cos y$. This ODE has the particular solution $b_2^p(y) = -\frac{1}{5}\cos y$. Adding a solution of the homogeneous ODE, we obtain $b_2 = a\cosh[2(y-L)] + b\sinh[2(y-L)] - \frac{1}{5}\cos y$. The boundary condition at $y = L$ gives $a - 1/5\cos L = 0$ hence $a = 1/5\cos L$. The boundary condition at $y = 0$ gives $f_2 = \frac{1}{5}[\cos L\cosh(2L) - 1] - b\sinh(2L)$. Then

$$b_2(y) = \frac{\frac{1}{5}[\cos L\cosh(2L) - 1] - f_2}{\sinh(2L)}\sinh[2(y-L)] + \frac{\cosh[2(y-L)] - \cos y}{5}.$$

We find

$$u(x,y) = \left(\frac{\frac{1}{5}[\cos L\cosh(2L) - 1] - f_2}{\sinh(2L)}\sinh[2(y-L)] + \frac{\cosh[2(y-L)] - \cos y}{5}\right)\sin(2x)$$
$$-\frac{f_1}{\sinh L}\sin x\sinh(y-L) - \sum_{n=3}^{\infty} f_n \frac{\sinh[n(y-L)]}{\sinh(nL)}\sin(nx).$$

AEROSPACE ENGINEERING.
Advanced Mathematics. Midterm exam. November 18th, 2013.
Calculators, phones, tablets not allowed

Select and solve four out of the following five problems.

1. The number of compulsive gamblers in the Madrid region increases at a rate proportional to their current population $y(t)$. Their growth rate is proportional to $1-y/C$, where the carrying capacity C of potential Eurovegas customers is three million. In an ideal situation, when there are very few gamblers compared to C, their number duplicates every three months in the absence of any other factors. Once the casinos are fully operational, the ordinary action of local parishes, psychotherapists and other philanthropic types may wean a thousand suckers off gambling per month. With an estimated ten thousand initial inveterate gamblers, these efforts come to nothing. Calculate the minimum number of repentant sinners they need to produce every month (possibly helped by a change in the tax code) to drive gambling to extinction.

   Solution. The gambling population y evolves according to the ODE:

$$\frac{dy}{dt} = ry\left(1 - \frac{y}{C}\right) - K,$$

   where K is the number of repentant gamblers per month. In the absence of any other factors, $dy/dt \approx ry$ gives $y(t) = y_0 e^{rt}$. Duplication every three months mean $2y_0 = y_0 e^{3r}$ or $r = \frac{1}{3}\ln 2$ per month. With $K = C/3000$ (1000 gamblers per month), the gambling population has two real-valued equilibria ($dy/dt = 0$) $y_\pm = \frac{C}{2} \pm \sqrt{\frac{C^2}{4} - \frac{CK}{r}}$. Notice that $dy/dt < 0$ for $y < y_-$ and $y > y_+$, whereas $dy/dt > 0$ for $y_- < y < y_+$. For all initial conditions $y_0 > y_-$, the number of gamblers tends to the other equilibrium point y_+, whereas they are driven to extinction in a virtuous society where $y_0 < y_-$. In our example, $y_- \approx 4334$ which is certainly a small number (close to $K/r \approx 4328$) as compared to $y_0 = 10,000$. To eradicate the gambling vice, the righteous crowd should achieve $K_c = rC/4 \approx 173,287$ conversions per month or more, so that $y_+ = y_- = C/2$ (corresponding to $f(y) = f'(y) = 0$ with the maximum of the parabola $f(y) = ry(1 - y/C) - K = \frac{dy}{dt}$ equal to 0) and $dy/dt < 0$ for any larger value $K > K_c$.

2. Classify the fixed point of the following linear system:

$$\begin{cases} \frac{dx}{dt} = 4x - y, \\ \frac{dy}{dt} = 2x + y. \end{cases}$$

Find the general solution of the system and solve it for the initial condition $(x_0, y_0) = (3, 4)$.

Solution. The characteristic equation is $\lambda^2 - 5\lambda + 6 = 0$ *and the eigenvalues are 2, with eigenvector* $(1, 2)$*, and 3, with eigenvector* $(1, 1)$*. The origin is an unstable node. The general solution of the system is*

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t}.$$

For the given initial condition, we find $c_1 = 1$*,* $c_2 = 2$*, hence* $x(t) = e^{2t} + 2e^{3t}$*,* $y(t) = 2e^{2t} + 2e^{3t}$*.*

3. Find the general solution of the system of ODEs

$$\begin{cases} \frac{dx}{dt} = -3x + 4y + \cos t, \\ \frac{dy}{dt} = -2x + 3y + 1. \end{cases}$$

Solution. We know that the eigenvalues are ± 1*, with eigenvectors* $(1, 1)$ *and* $(2, 1)$*, respectively. The fundamental matrix is therefore*

$$\underline{\underline{\Psi}} = \begin{pmatrix} e^t & 2e^{-t} \\ e^t & e^{-t} \end{pmatrix}.$$

Its inverse is

$$\underline{\underline{\Psi}}^{-1} = \begin{pmatrix} -e^{-t} & 2e^{-t} \\ e^t & -e^t \end{pmatrix}.$$

Then

$$\int \underline{\underline{\Psi}}^{-1} \underline{F}(t) dt = \int \begin{pmatrix} e^{-t}(2 - \cos t) \\ e^t(\cos t - 1) \end{pmatrix} dt = \begin{pmatrix} -e^{-t}[2 + \frac{1}{2}(\sin t - \cos t)] \\ e^t[\frac{1}{2}(\cos t + \sin t) - 1] \end{pmatrix}.$$

Using now

$$\underline{x}(t) = \underline{\underline{\Psi}}(t) \int \underline{\underline{\Psi}}^{-1} \underline{F}(t) dt = \begin{pmatrix} -4 + \frac{1}{2}\sin t + \frac{3}{2}\cos t \\ -3 + \cos t \end{pmatrix},$$

we find a particular solution to which we add the general solution of the homogeneous system:

$$\underline{x}(t) = \underline{\underline{\Psi}}(t) \int \underline{\underline{\Psi}}^{-1} \underline{F}(t) dt = \begin{pmatrix} -4 + \frac{1}{2}\sin t + \frac{3}{2}\cos t + c_1 e^t + 2c_2 e^{-t} \\ -3 + \cos t + c_1 e^t + c_2 e^{-t} \end{pmatrix}.$$

The method of undetermined coefficients would give $\underline{x}_p = \underline{a} + \underline{b}\cos t + \underline{c}\sin t$ *which, inserted in the system produces* $-\underline{\underline{A}}\underline{a} = (0, 1)$*,* $\underline{c} - \underline{\underline{A}}\underline{b} = (1, 0)$ *and* $-\underline{b} = \underline{\underline{A}}\underline{c}$*. Since* $\underline{\underline{A}}^2 = \underline{\underline{I}}$*,* $\underline{a} = -\underline{\underline{A}}(0, 1) = (-4, -3)$ *and* $\underline{c} = -\underline{\underline{A}}\underline{b}$*. Then* $2\underline{c} = (1, 0)$ *and* $\underline{c} = \frac{1}{2}(1, 0)$*,* $\underline{b} = -\frac{1}{2}\underline{\underline{A}}(1, 0) = (\frac{3}{2}, 1)$*. This yields the same particular solution as before.*

4. Find the eigenvalues and eigenfunctions of the boundary value problem: $u'' + 4u' + 3u + \lambda u = 0$, $u(0) = 0 = u(1)$. Write it as a Sturm-Liouville eigenvalue problem: $[p(x)\phi']' + q(x)\phi + \lambda\sigma(x)\phi = 0$, determine p, q and σ and write the orthogonality condition for the eigenfunctions.

   Solution. Setting $u = e^{rx}$, we get $r^2 + 4r + \lambda + 3 = 0$, that gives $r = -2 \pm i\sqrt{\lambda - 1}$. Then $u = e^{-2x}[c_1 \cos(\sqrt{\lambda - 1}x) + c_2 \sin(\sqrt{\lambda - 1}x)]$. The boundary conditions yield $c_1 = 0$ and $\sin\sqrt{\lambda - 1} = 0$. Therefore ($\sqrt{\lambda - 1} = n\pi$ with $n = 1, 2, \ldots$ and we have obtained the eigenvalues $\lambda_n = 1 + n^2\pi^2$, $n = 1, 2, \ldots$ and the eigenfunctions $\phi_n(x) = e^{-2x}\sin(n\pi x)$. To transform into a Sturm-Liouville ODE, we multiply the original equation by e^{4x}, thereby obtaining $(e^{4x}u')' + (\lambda + 3)e^{4x}u = 0$. We identify: $p(x) = e^{4x}$, $q(x) = 3e^{4x}$ and $\sigma(x) = e^{4x}$. The orthogonality condition for the eigenfunctions is $\int_0^1 e^{4x}\phi_m(x)\phi_n(x)\,dx = 0$ (i.e., $\int_0^1 \sin m\pi x \sin n\pi x\,dx = 0$) if $m \neq n$.

5. Find the Fourier sine series of $f(x) = 1 - x$ for $0 < x < 1$. What is the Fourier cosine series of $f'(x)$ with the same period?

   Solution. The Fourier coefficients are

$$b_n = 2\int_0^1 (1 - x)\sin n\pi x\,dx = -\frac{2}{n\pi}(1 - x)\cos n\pi x|_0^1 - \frac{2}{n\pi}\int_0^1 \cos n\pi x\,dx = \frac{2}{n\pi}.$$

   The Fourier sine series is therefore

$$\sum_{n=1}^{\infty} \frac{2}{n\pi}\sin n\pi x.$$

   The odd extension of $f(x)$ has a discontinuity at $x = 0$ therefore it is not possible to differentiate the previous series term by term. However, $f'(x) = -1$ in $(0, 1)$ has the even extension $f'(x) = -1$ to $(-1, 0)$ and the corresponding Fourier series is simply -1.

AEROSPACE ENGINEERING.
Advanced Mathematics. Exam June 17th, 2014.

Select and solve four out of the following five problems (2.5/10 each, 3 hours total time).

1. Suppose you are able to find three linearly independent particular solutions $u_1(x)$, $u_2(x)$ and $u_3(x)$ of the second order linear differential equation $L[y] = f(x)$. What is the general solution?

   Solution. $y(x) = u_1(x) + c_1[u_2(x) - u_1(x)] + c_2[u_3(x) - u_1(x)]$, c_1, c_2 *arbitrary numbers. In fact,* $L[u_2 - u_1] = L[u_2] - L[u_1] = f - f = 0$ *because of the linearity of* L. *Then* $u_2 - u_1$ *and, similarly,* $u_3 - u_1$, *are independent solutions of the homogeneous equation* $L[y] = 0$.

2. Classify the fixed point of the following linear system:
   $\dot{x} = x + y$, $\dot{y} = -x - 2y$,
   and find its general solution.

   Solution. The eigenvalue problem is $\lambda^2 + \lambda - 1 = 0$ *which has solutions* $\frac{1}{2}(-1 \pm \sqrt{5})$ $\lambda_- < 0 < \lambda_+$*) and therefore the origin is a saddle point. The eigenvectors are* $\underline{v}_\pm = (1, \frac{1}{2}(-3 \pm \sqrt{5}))'$. *The general solution is* $\underline{x}(t) = c_-(1, \frac{\sqrt{5}-3}{2})'e^{(-1+\sqrt{5})t/2} + c_+(1, -\frac{\sqrt{5}+3}{2})'e^{-(1+\sqrt{5})t/2}$.

3. A cylindrical slender rod is made out of two metals with positive thermal conductivities, specific heats and mass densities K_1, c_1 and ρ_1 for $0 < x < a$ and K_2, c_2 and ρ_2 for $a < x < L$. Solve the homogeneous heat equation

$$\rho(x)c(x)\frac{\partial u}{\partial t} = K(x)\frac{\partial^2 u}{\partial x^2},$$

   with boundary and initial conditions $u(0,t) = 0$, $u(L,t) = 0$, $u(x,0) = 1$. Recall that the heat flux is continuous throughout the rod.

   Solution. Separating variables, $u = X(x)T(t)$, *we find*

$$\frac{T''}{T} = \frac{K(x)}{\rho(x)c(x)}\frac{X''}{X} = -\lambda.$$

   The eigenvalue problem for X *is*

$$X'' + \lambda\frac{\rho_1 c_1}{K_1}X = 0, \quad 0 < x < a,$$
$$X'' + \lambda\frac{\rho_2 c_2}{K_2}X = 0, \quad a < x < L,$$

with $X(0) = 0$, $X(L) = 0$. The solution of this Sturm-Liouville problem with weight function $w(x) = \rho(x)c(x)/K(x)$ is

$$X(x) = \begin{cases} \sin(r_1 x), & 0 < x < a, \\ \mu \sin[(r_2(L-x)], & a < x < L, \end{cases}$$

where $r_i = \sqrt{\lambda \rho_i c_i / K_i}$, $i = 1, 2$ and μ is a constant. Continuity of X and of the flux $-K_i X'(x)$ at $x = a$ yield

$$\sin(r_1 a) = \mu \sin[r_2(L-a)], \quad K_1 r_1 \cos(r_1 a) = -K_2 r_2 \mu \cos[r_2(L-a)].$$

Elimination of μ produces the eigenvalue equation

$$\tan(r_1 a) = -\frac{K_1 r_1}{K_2 r_2} \tan[r_2(L-a)] \Longrightarrow \tan(r_1 a) = -\sqrt{\frac{K_1 \rho_1 c_1}{K_2 \rho_2 c_2}} \tan[r_2(L-a)].$$

This equation has infinitely many solutions $\lambda_n > 0$, $n = 1, 2, \ldots$ that increase with n. Let $X_n(x)$ be the corresponding eigenfunctions. Their orthogonality condition is

$$\int_0^L w(x) X_n(x) X_m(x) dx = 0, \quad n \neq m; \quad w(x) = \frac{\rho(x)c(x)}{K(x)}.$$

The general solution is $u(x,t) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n t} X_n(x)$. From the initial condition, we obtain

$$1 = \sum_{n=1}^{\infty} a_n X_n(x) \Longrightarrow a_n = \frac{\int_0^L w(x) X_n(x) dx}{\int_0^L w(x) X_n^2(x) dx} \Longrightarrow$$
$$a_n = 4 \frac{\sqrt{\frac{w_1}{\lambda_n}}[1 - \cos(r_1 a)] + \sqrt{\frac{w_2}{\lambda_n}}[\cos(r_2(L-a)) - 1]}{2w_1 a + 2w_2(L-a) - \sqrt{\frac{w_1}{\lambda_n}} \sin(2r_1 a) - \sqrt{\frac{w_2}{\lambda_n}} \sin(2r_2(L-a))}.$$

4. Write a finite difference scheme to solve the telegraph equation:

$$\frac{\partial^2 u}{\partial t^2} + a \frac{\partial u}{\partial t} - b\, u = \frac{\partial^2 u}{\partial x^2},$$

using centered differences to approximate all differentials. Find the CFL condition (ignoring boundary conditions and for a sufficiently small product of the time step times $|a|$).

Solution. $u_j^{n+1} + u_j^{n-1} - 2u_j^n - s\,(u_{j+1}^n + u_{j-1}^n - 2u_j^n) + a\,\Delta t\,(u_j^{n+1} - u_j^{n-1})/2 - b u_j^n (\Delta t)^2 = 0$, *with* $s = (\Delta t/\Delta x)^2$.
The CFL condition is found by substituting $u_j^n = e^{ij\alpha} Q^n$ in the difference equation. We find

$$\left(1 + \frac{a\Delta t}{2}\right) Q^2 - 2\left[1 - s(1 - \cos\alpha) + \frac{b(\Delta t)^2}{2}\right] Q + 1 - \frac{a\Delta t}{2} = 0.$$

The solutions are complex conjugate and have modulus smaller than 1 provided: $0 < a\,\Delta t < 2$ *and*

$$\left[1 - s(1-\cos\alpha) + \frac{b(\Delta t)^2}{2}\right]^2 < \left(1 + \frac{a\Delta t}{2}\right)\left(1 - \frac{a\Delta t}{2}\right) = 1 - \left(\frac{a\Delta t}{2}\right)^2,$$

or, equivalently,

$$-\sqrt{1 - \frac{a^2(\Delta t)^2}{4}} < 1 - 2s\sin^2\frac{\alpha}{2} + b(\Delta t)^2/2 < \sqrt{1 - \frac{a^2(\Delta t)^2}{4}}.$$

This holds if

$$s\sin^2\frac{\alpha}{2} < \frac{1}{2} + \frac{b(\Delta t)^2}{4} + \frac{1}{2}\sqrt{1 - \frac{a^2(\Delta t)^2}{4}} = 1 + \frac{b(\Delta t)^2}{4} - \frac{1}{2}\left(1 - \sqrt{1 - \frac{a^2(\Delta t)^2}{4}}\right),$$

i.e.,

$$s\sin^2\frac{\alpha}{2} < 1 + \frac{b(\Delta t)^2}{4} - \frac{a^2(\Delta t)^2}{8}\frac{1}{1 + \sqrt{1 - \frac{a^2(\Delta t)^2}{4}}}.$$

From this inequality and from $|sin(\alpha/2)| < 1$ *and* $-1 < -1/(1+B)$*, we find the CFL condition*

$$s < 1 - \frac{(\Delta t)^2(a^2 - 2b)}{8} \Leftrightarrow \frac{\Delta x}{\Delta t} > \frac{1}{\sqrt{1 - (a^2 - 2b)\frac{(\Delta t)^2}{8}}}.$$

Here we must have $0 < (a^2 - 2b)\frac{(\Delta t)^2}{8} < 1$*, which places another restriction on the coefficients* a*,* b *and the time step* Δt*.*

5. Solve the following IBVP:

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= c^2\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L,\, t > 0, \\ u(x,0) &= \frac{\partial u}{\partial t}(x,0) = 0, \quad 0 < x < L, \\ u(0,t) &= h(t), \quad u(L,t) = 0, \quad t > 0. \end{aligned}$$

Solution. We try the eigenfunction expansion of the solution, $u(x,t) = \sum_{n=1}^{\infty} a_n(t)\sin\frac{n\pi x}{L}$*. Using*

$$\int_0^L \frac{\partial^2 u}{\partial x^2}\sin\frac{n\pi x}{L}\,dx = -\frac{n\pi}{L}u\cos\frac{n\pi x}{L}\Big|_0^L - \frac{n^2\pi^2}{L^2}\int_0^L u\sin\frac{n\pi x}{L}\,dx,$$

we get

$$\ddot{a}_n + \frac{n^2\pi^2c^2}{L^2}a_n = \frac{2n\pi c^2}{L^2}h(t), \quad a_n(0) = \dot{a}_n(0) = 0.$$

The solution of this IVP can be found by variation of parameters or any other method as

$$a_n(t) = \frac{2c}{L}\int_0^t h(s)\sin\frac{n\pi c(t-s)}{L}\,ds.$$

Then

$$u(x,t) = \frac{2c}{L}\sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}\int_0^t h(s)\sin\frac{n\pi c(t-s)}{L}\,ds.$$

It is also possible to separate, $u(x,t) = h(t)(1-x/L)+v(x,t)$*, and solve the following IBVP with homogeneous b.c.:*

$$\begin{aligned}
&\frac{\partial^2 v}{\partial t^2} - c^2\frac{\partial^2 v}{\partial x^2} = -\Big(1-\frac{x}{L}\Big)\ddot{h}, \quad 0<x<L,\ t>0,\\
&v(x,0) = -\Big(1-\frac{x}{L}\Big)h(0),\ \frac{\partial v}{\partial t}(x,0) = -\Big(1-\frac{x}{L}\Big)\dot{h}(0), \quad 0<x<L,\\
&v(0,t) = 0, \quad v(L,t) = 0, \quad t>0.
\end{aligned}$$

By eigenfunction expansion, we find

$$\ddot{b}_n + \frac{n^2\pi^2c^2}{L^2}b_n = -\frac{2}{n\pi}\ddot{h}(t), \quad b_n(0) = -\frac{2}{n\pi}h(0),\ \dot{b}_n(0) = -\frac{2}{n\pi}\dot{h}(0),$$

for the Fourier coefficients of $v(x,t)$*. Then* $v(x,t) = \sum_{n=1}^{\infty} b_n(t)\sin\frac{n\pi x}{L}$*, with*

$$b_n(t) = -\frac{4c}{n\pi L}\int_0^t \ddot{h}(s)\sin\frac{n\pi c(t-s)}{L}\,ds - \frac{2}{n\pi}h(0)\cos\frac{n\pi ct}{L} - \frac{2L}{n^2\pi^2c}\dot{h}(0)\sin\frac{n\pi ct}{L}.$$

Yet another way to solve this problem is to use the method of characteristics taking into account the infinitely many reflections at the boundaries.

AEROSPACE ENGINEERING.
Advanced Mathematics. Final exam. January 23rd, 2014.
Computers, calculators, phones, tablets not allowed

Select and solve four out of the following five problems.

1. A string $(0, L)$ consists of two homogeneous parts of equal cross sections, smoothly joined at the point $x = x_0$, and having velocities, Young moduli (= tensions) and mass densities c_1, E_1, ρ_1 and c_2, E_2 and ρ_2, respectively, with $E_2 > E_1$, $\rho_2 > \rho_1$, $c_i^2 = E_i/\rho_i$. Find the displacement in such a string if both ends are fixed, $u(x,0) = x/x_0$ for $0 < x < x_0$, $u(x,0) = (L-x)/(L-x_0)$ for $x_0 < x < L$, and $u_t(x,0) = \partial u(x,0)/\partial t = 0$. *Hint.* u *and* $E(x)u_x$ *are continuous at* $x = x_0$ *with* $E(x) = E_1$ *for* $x < x_0$ *and* $E(x) = E_2$ *for* $x > x_0$.

Solution. We have to solve the IBVP:

$$\begin{aligned}
\rho_1 \partial_t^2 u = E_1 \partial_x^2 u, \quad 0 < x < x_0, \quad t > 0,\\
\rho_2 \partial_t^2 u = E_2 \partial_x^2 u, \quad x_0 < x < L, \quad t > 0,\\
u(x_0-,t) = u(x_0+,t), \quad E_1 \partial_x u(x_0-,t) = E_2 \partial_x u(x_0+,t),\\
u(0,t) = u(L,t) = 0,\\
u(x,0) = \begin{cases} \frac{x}{x_0}, & 0 < x < x_0,\\ \frac{L-x}{L-x_0}, & x_0 < x < L, \end{cases} \qquad \partial_t u(x,0) = 0.
\end{aligned}$$

The wave equation can be written as

$$\partial_t^2 u = [c(x)]^2 \partial_x^2 u, \quad c(x) = \begin{cases} c_1, & 0 < x < x_0,\\ c_2, & x_0 < x < L, \end{cases}$$

where $c_i^2 = E_i/\rho_i$. *By separation of variables,* $u(x,t) = X(x)T(t)$, *we get*

$$\frac{T''(t)}{T(t)} = [c(x)]^2 \frac{X''(x)}{X(x)} = -\lambda.$$

Using now $\lambda = \omega^2$, *we obtain:*

$$\begin{aligned}
&T'' + \omega^2 T = 0, t > 0,\\
&X'' + \frac{\omega^2}{c_1^2} X = 0, \quad 0 < x < x_0,
\end{aligned}$$

$$
\begin{aligned}
&X'' + \frac{\omega^2}{c_2^2} X = 0, \quad x_0 < x < L, \\
&X(x_0-) = X(x_0+), \quad E_1 X'(x_0-) = E_2 X'(x_0+), \\
&X(0) = 0 = X(L).
\end{aligned}
$$

The BVP is a Sturm-Liouville eigenvalue problem with eigenvalue ω^2 and weight function $1/c_1^2$ for $0 < x < x_0$ and $1/c_2^2$ for $x_0 < x < L$. The solution of the BVP is

$$
X(x) = \begin{cases} a \sin k_1 x, & 0 < x < x_0, \\ b \sin[k_2(x-L)], & x_0 < x < L, \end{cases}
$$

with $k_i = \omega/c_i$, $c_i = \sqrt{E_i/\rho_i}$. Continuity of X and $E_i X'$ at $x = x_0$ yields

$$
\begin{aligned}
a \sin k_1 x_0 &= b \sin[k_2(x_0 - L)], \\
a E_1 k_1 \cos k_1 x_0 &= b E_2 k_2 \cos[k_2(x_0 - L)],
\end{aligned}
$$

which produces the eigenvalue problem

$$
\tan \frac{\omega}{c_1} x_0 = \sqrt{\frac{E_1 \rho_1}{E_2 \rho_2}} \tan[\frac{\omega}{c_2}(x_0 - L)], \quad \frac{E_1 k_1}{E_2 k_2} = \sqrt{\frac{E_1 \rho_1}{E_2 \rho_2}}.
$$

There are infinitely many solutions $\omega_1 < \omega_2 < \ldots$. To each it corresponds the eigenfunction

$$
X_n(x) = \begin{cases} \frac{\sin \frac{\omega_n}{c_1} x}{\sin \frac{\omega_n}{c_1} x_0}, & 0 < x < x_0, \\ \frac{\sin\left[\frac{\omega_n}{c_2}(x-L)\right]}{\sin\left[\frac{\omega_n}{c_2}(x_0-L)\right]}, & x_0 < x < L. \end{cases}
$$

We have

$$
\begin{aligned}
\|X_n\|^2 &= \frac{\omega_n^2}{c_1^2} \int_0^{x_0} \sin k_1 x_0 \frac{\sin^2 \frac{\omega_n}{c_1} x}{\sin^2 \frac{\omega_n}{c_1} x_0} dx + \frac{\omega_n^2}{c_2^2} \int_{x_0}^{L} \frac{\sin^2\left[\frac{\omega_n}{c_2}(x-L)\right]}{\sin^2\left[\frac{\omega_n}{c_2}(x_0-L)\right]} dx \\
&= \frac{\omega_n^2 x_0}{2c_1^2 \sin^2 \frac{\omega_n}{c_1} x_0} + \frac{\omega_n^2 (L - x_0)}{2c_2^2 \sin^2\left[\frac{\omega_n}{c_2}(x_0 - L)\right]}.
\end{aligned}
$$

The solution that satisfies the initial conditions is

$$
\begin{aligned}
u(x,t) &= \sum_{n=1}^{\infty} a_n X_n(x) \cos \omega_n t, \\
a_n &= \frac{1}{c_1^2 \|X_n\|^2} \int_0^{x_0} u(x,0) X_n(x) dx + \frac{1}{c_2^2 \|X_n\|^2} \int_{x_0}^{L} u(x,0) X_n(x) dx \\
&= \frac{1}{\|X_n\|^2} \left(\frac{E_1}{x_0} + \frac{E_2}{L - x_0} \right).
\end{aligned}
$$

2. Consider the first order hyperbolic PDE $u_t + cu_x = x$ for $u(x,t)$, where subscripts mean partial derivatives. Approximate u_t and u_x by backward finite differences and by centered finite differences, respectively. Calculate the truncation error and the stability of the resulting numerical scheme.

Solution. The numerical scheme is

$$\frac{u_j^{(m)} - u_j^{(m-1)}}{\Delta t} + \frac{c}{2\Delta x}(u_{j+1}^{(m)} - u_{j-1}^{(m)}) = j\Delta x.$$

The truncation error is $\tau_j^{(m)}(\Delta t, \Delta x) = \frac{u_j^{(m)} - u_j^{(m-1)}}{\Delta t} + \frac{c}{2\Delta x}(u_{j+1}^{(m)} - u_{j-1}^{(m)}) - j\Delta x$, *where* $u_j^{(m)}$ *should be replaced by* $u(j(\Delta x), m(\Delta t))$ *and* $x = j(\Delta x)$, $t = m(\Delta t)$. *We find*

$$\tau_j^{(m)}(\Delta t, \Delta x) = \frac{u(x,t) - u(x,t-\Delta t)}{\Delta t} + \frac{c}{2\Delta x}[u(x+\Delta x, t) - u(x-\Delta x, t)] - x$$
$$= u_t + \frac{\Delta t}{2}u_{tt} + O((\Delta t)^2) + cu_x + \frac{c(\Delta x)^2}{6}u_{xxx} + O((\Delta x)^4) - x = O(\Delta t) + O((\Delta x)^2),$$

after using the PDE $u_t + cu_x = x$. *To ascertain the stability of the scheme, we substitute in it* $u_j^{(m)} = Q^m e^{ij\alpha\Delta x}$ *and keep the terms corresponding to the highest-order differences, thereby getting*

$$1 - \frac{1}{Q} + is\sin(\alpha\Delta x) = 0, \quad s = \frac{c\Delta t}{\Delta x}.$$

We find $Q^{-1} = 1 + is\sin(\alpha\Delta x)$, *and therefore*

$$|Q|^{-2} = 1 + s^2\sin^2(\alpha\Delta x) \geq 1 \Longrightarrow |Q| \leq 1,$$

and the scheme is stable for any real value of s.

3. Solve the wave equation $u_{tt} = c^2 u_{xx}$ on $x > 0$, $t > 0$, subject to the boundary condition $u_x(0,t) + hu(0,t) = q(t)$, $t > 0$, with $q(0) = 0$, and initial conditions $u(x,0) = u_t(x,0) = 0$ for $x > 0$.

Solution. Clearly $u = 0$ *for* $x > ct$ *due to the initial conditions. Then* $u = F(x-ct) + G(x+ct)$ *gives* $F(x) = G(x) = 0$ *for* $x > 0$. *For* $x < ct$, $u = F(x-ct)$ *(with* $x - ct < 0$*) which is not zero. Inserting in the boundary condition, we obtain*

$$F'(-ct) + hF(-ct) = q(t) \Longrightarrow F'(\xi) + hF(\xi) = q(-\xi/c) \quad \text{for } \xi = -ct.$$

This is a first order ODE which can be solved easily multiplying it by the integrating factor $e^{h\xi}$. *The solution is*

$$F(\xi) = \int_0^\xi e^{-h(\xi-s)}q(-s/c)\,ds = -c\int_0^{-\xi/c} e^{-h(\xi+c\tau)}q(\tau)d\tau,$$

and

$$u(x,t) = F(x-ct) = -c\int_0^{t-x/c} e^{-h(x-ct+c\tau)}q(\tau)d\tau, \quad x < ct.$$

4. Solve the exterior Dirichlet problem $\nabla^2 u = 0$ (in polar coordinates) on $1 < r < \infty$ for the boundary conditions $u(1,\theta) = 1$ for $0 \le \theta < \pi$, $u(1,\theta) = 0$ for $\pi \le \theta < 2\pi$, $|u(\infty,\theta)| < \infty$.

Solution. Separation of variables for $u_{rr} + r^{-1}u_r + r^{-2}u_{\theta\theta} = 0$ *produces* $r^2R'' + rR' - \lambda R = 0$ *and* $\Theta'' + \lambda\Theta = 0$ *if* $u = R(r)\Theta(\theta)$. *As usual* $\lambda = n^2$ *for eigenfunctions* $\cos n\theta$ *and* $\sin n\theta$. *The solutions of the corresponding Euler equations for* R *are* $R_n = a_n r^n + b_n/r^n$ *and* $R_0 = a_0 \ln r + b_0$. *Clearly solutions that are bounded at* $r = \infty$ *correspond to having* $a_n = 0$ *for all* $n \ge 0$. *Then the superposition principle yields*

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} \frac{1}{r^n}(A_n \cos n\theta + B_n \sin n\theta).$$

The boundary condition yields

$$\begin{aligned} A_0 &= \frac{1}{2\pi}\int_0^{2\pi} u(1,\theta)d\theta = \frac{1}{2}, \\ A_n &= \frac{1}{\pi}\int_0^{2\pi} u(1,\theta)\cos n\theta\, d\theta = \frac{1}{\pi}\int_0^{\pi} \cos n\theta\, d\theta = 0, \\ B_n &= \frac{1}{\pi}\int_0^{2\pi} u(1,\theta)\sin n\theta\, d\theta = \frac{1}{\pi}\int_0^{\pi} \sin n\theta\, d\theta = \frac{1-(-1)^n}{n\pi}. \end{aligned}$$

Then

$$u(r,\theta) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{(2n-1)\pi r^{2n-1}} \sin[(2n-1)\theta].$$

5. Solve Laplace's equation on the triangular region $\{(x,y), 0 < x < 1, 0 < y < x\}$ with boundary conditions $u(x,0) = x$ for $0 < x < 1$, $u(1,y) = 0$ for $0 < y < 1$ and $u(x,x) = 0$ for $0 < x < 1$. *Hint: Show that the solution of this problem is the same as the solution of Laplace's equation on the unit square with additional boundary conditions* $u(x,1) = 0$ *for* $0 < x < 1$ *and* $u(0,y) = -y$ *for* $0 < y < 1$ *(antisymmetric data with respect to the diagonal* $y = x$*).*

Solution. According to the hint, the solution on the square is antisymmetric with respect to the diagonal $y = x$, *so that* $u(y,x) = -u(x,y)$ *and therefore* $u(x,x) = 0$. *The solution on the unit square is split into two separate problems:* $u = u^I(x,y) + u^{II}(x,y)$, *where both these functions solve Laplace's equation with different boundary conditions: (i)* $u^I = 0$ *on* $x = 0,\ 1$ *and* $y = 1$, *and* $u^I = x$ *on* $y = 0$*; (ii)* $u^{II} = 0$ *on* $y = 0,\ 1$ *and* $x = 1$, *and* $u^{II} = -y$ *on* $x = 0$. *The solution of problem (i) is* $u^I = \sum_{n=1}^{\infty} a_n(y)\sin n\pi x$, *which inserted into Laplace's equation produces* $a_n'' - n^2\pi^2 a_n = 0$ *with* $a_n(1) = 0$. *The solution of this problem is* $a_n(y) = A_n \sinh[n\pi(y-1)]$. *The leftover boundary condition yields* $x = u^I(x,0) = -\sum_{n=1}^{\infty} A_n \sinh(n\pi)\sin n\pi x$. *Integration by parts produces*

$$\begin{aligned} A_n &= -\frac{2}{\sinh n\pi}\int_0^1 x\sin n\pi x\, dx = \frac{2x}{n\pi\sinh n\pi}\cos n\pi x|_0^1 - \frac{2}{n\pi\sinh n\pi}\int_0^1 \cos n\pi x\, dx \\ &= (-1)^n \frac{2}{n\pi\sinh n\pi}. \end{aligned}$$

Then

$$u^I(x,y) = \sum_{n=1}^{\infty} (-1)^n \frac{2}{n\pi \sinh n\pi} \sin n\pi x \sinh[n\pi(y-1)].$$

Similarly $u^{II}(x,y) = -u^I(y,x)$*:*

$$u^{II}(x,y) = -\sum_{n=1}^{\infty} (-1)^n \frac{2}{n\pi \sinh n\pi} \sin n\pi y \sinh[n\pi(x-1)],$$

and therefore the solution of the BVP is

$$u(x,y) = \sum_{n=1}^{\infty} \frac{2(-1)^n}{n\pi \sinh n\pi} \{\sin(n\pi x) \sinh[n\pi(y-1)] - \sin(n\pi y) \sinh[n\pi(x-1)]\}.$$

AEROSPACE ENGINEERING.
Advanced Mathematics. Final exam 06/25/2013, solutions.

Select and solve four out of the following five problems.

1. Answer each of the following questions about the initial value problem (IVP):
$y'' - 4y' + 29y = 0$; $y(0) = -3$, $y'(0) = 4$.

   (a) Convert the IVP into a system of differential equations and write your answer in matrix form.

   (b) Solve the system and use this solution to find the solution to the original 2nd order IVP.

   (c) Sketch the phase portrait for the system and determine the stability of the equilibrium solution.

Solution. First convert to a system in matrix form: $y' = z$, $z' = -29y + 4z$ *which gives*

$$\underline{x}' = \begin{pmatrix} 0 & 1 \\ -29 & 4 \end{pmatrix} \underline{x}, \quad \underline{x}(0) = \begin{pmatrix} -3 \\ 4 \end{pmatrix}.$$

The eigenvalues and eigenvectors for the coefficient matrix are

$$\lambda = 2 + 5i, \quad \underline{\psi} = \begin{pmatrix} 1 \\ 2+5i \end{pmatrix},$$

and their complex conjugates. Then

$$\begin{aligned} e^{(2+5i)t} \begin{pmatrix} 1 \\ -2+5i \end{pmatrix} &= e^{2t}(\cos 5t + i \sin 5t) \begin{pmatrix} 1 \\ -2+5i \end{pmatrix} \\ &= e^{2t} \begin{pmatrix} \cos 5t \\ 2\cos 5t - 5 \sin 5t \end{pmatrix} + ie^{2t} \begin{pmatrix} \sin 5t \\ 5\cos 5t + 2 \sin 5t \end{pmatrix}, \end{aligned}$$

and the general solution of the system is

$$\underline{x} = c_1 e^{2t} \begin{pmatrix} \cos 5t \\ 2\cos 5t - 5 \sin 5t \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} \sin 5t \\ 5\cos 5t + 2 \sin 5t \end{pmatrix}.$$

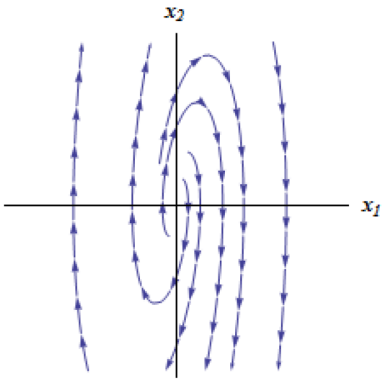
Using the initial condition we find the system of equations

$$c_1 = -3, \quad 2c_1 + 5c_2 = 4 \Longrightarrow c_2 = 2.$$

The solution is then

$$\underline{x}(t) = -3e^{2t} \begin{pmatrix} \cos 5t \\ 2\cos 5t - 5 \sin 5t \end{pmatrix} + 2e^{2t} \begin{pmatrix} \sin 5t \\ 5\cos 5t + 2 \sin 5t \end{pmatrix} = e^{2t} \begin{pmatrix} 2\sin 5t - 3\cos 5t \\ 4\cos 5t + 19 \sin 5t \end{pmatrix}.$$

The solution to the original differential equation is the top row, $y(t) = e^{2t}(-3\cos 5t + 2\sin 5t)$. The velocity vector $\underline{x}'$ corresponding to $(1,0)$ is obtained from the system as $\underline{x}' = (0,-29)$ and it points vertically downwards. We can see that trajectories will cross the x_1 axis going downwards. A sketch of the phase portrait is below. The equilibrium solution in this case is an unstable spiral.



2. A population of fish in a lake grows at a rate proportional to its population. There are originally 600 fish in the lake and in the absence of any outside factors the population will double in 4 months time (assume 4 weeks/month). Each week there is a net migration of 30 fish into the lake and predators eat 47 fish. Derive an ODE for the fish population growth per month.

   (a) Will the fish population survive? If not when do they die out?

   (b) Two months after situation above starts something causes the population of the predators to increase and they will now eat 70 fish/week. The migration of fish into the lake does not change. Does the fish population survive this? If not when do they die out?

Solution. (a) The time unit is one month. Here are the IVPs for this problem:

$$\dot{P} = rP, \quad P(0) = 600,\ P(4) = 1200,$$
$$\dot{P}_1 = rP_1 + 30 \times 4 - 47 \times 4 = rP_1 - 68, \quad P_1(0) = 600.$$

The solution to the first one (so we can find r) is,

$$P(t) = 600e^{rt}, \quad 1200 = 600e^{4r} \Longrightarrow r = \frac{1}{4}\ln 2.$$

The solution to the second IVP is,

$$P_1(t) = \frac{272}{\ln 2} + 207.5869e^{t\ln 2/4}.$$

Both terms in this solution are positive and so the solution will never be zero. This in turn means that the population of fish will survive. (b) After 2 months, $P_1(2) = 685.9853$. The new equation for the fish is

$$\dot{P}_2 = rP_2 + (30 - 70) \times 4 = rP_2 - 160, \quad P_2(2) = P_1(2) = 685.9853.$$

The solution now is

$$P_2(t) = \frac{640}{\ln 2} - \left(\frac{640}{\ln 2} - 685.9853\right) e^{(t-2)\ln 2/4} = \frac{640}{\ln 2} - 237.3395\, e^{(t-2)\ln 2/4}.$$

Clearly the population will not survive and $P_2(t_f) = 0$ at $t_f - 2 \approx 7.84$. Thus the fish die out after 9.84 months (9 months and 25 days).

3. Solve the inhomogeneous heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - u + \sin t, \quad 0 < x < 1,$$

with boundary and initial conditions $u(0,t) = 0$, $u(1,t) = 0$, $u(x,0) = 0$.

Solution. Firstly, find a x-independent solution: $\dot{U} + U = \sin t$, namely, $U(t) = \frac{1}{2}(\sin t - \cos t)$. Rewrite the equation using $u(x,t) = U(t) + e^{-t}v(x,t)$. We find

$$v_t = v_{xx}, \quad v(0,t) = -\frac{e^t}{2}(\sin t - \cos t) = v(1,t), \quad v(x,0) = \frac{1}{2}.$$

As solution we try a Fourier sine series $v(x,t) = \sum_{n=1}^{\infty} b_n(t)\sin(n\pi x)$, thereby obtaining the equation

$$\dot{b}_n + n^2\pi^2 b_n = -\,2n\pi v(x,t)\cos(n\pi x)|_{x=0}^{1} = [1-(-1)^n]n\pi e^t(\cos t - \sin t).$$

The solution is

$$b_n(t) = \frac{[1-(-1)^n]n\pi e^t}{1+(1+n^2\pi^2)^2}[(2+n^2\pi^2)\cos t - n^2\pi^2 \sin t] + ce^{-n^2\pi^2 t}.$$

The initial condition yields

$$\frac{1-(-1)^n}{n\pi} = c + \frac{[1-(-1)^n]n\pi}{1+(1+n^2\pi^2)^2}(2+n^2\pi^2) \Longrightarrow c = \frac{1-(-1)^n}{n\pi}\frac{1+(1+\pi^2n^2)^2 - n^2\pi^2(2+n^2\pi^2)}{1+(1+n^2\pi^2)^2},$$

$$c = \frac{2[1-(-1)^n]}{n\pi[1+(1+n^2\pi^2)^2]}.$$

The overall solution is

$$u(x,t) = \frac{1}{2}(\sin t - \cos t) + \sum_{m=1}^{\infty} \frac{2}{1+(1+(2m-1)^2\pi^2)^2}\Big\{\frac{2e^{-t-(2m-1)^2\pi^2 t}}{(2m-1)\pi}$$
$$+n\pi[(2+(2m-1)^2\pi^2)\cos t - (2m-1)^2\pi^2 \sin t]\Big\}\sin[(2m-1)\pi x].$$

An equivalent solution is obtained by direct eigenfunction expansion of u: $u(x,t) = \sum_{n=1}^{\infty} B_n(t)\sin(n\pi x)$. We find

$$\dot{B}_n + (1+n^2\pi^2)B_n = 2\sin t \int_0^1 \sin(n\pi x)dx = \frac{2}{n\pi}[1-(-1)^n]\sin t,$$

which has to be solved with i.c. $B_n(0) = 0$. We find

$$B_n(t) = \frac{2}{n\pi}\frac{1-(-1)^n}{1+(1+n^2\pi^2)^2}[(1+n^2\pi^2)\sin t - \cos t] + C\,e^{-(1+n^2\pi^2)t},$$

and the i.c. yields

$$C = \frac{2}{n\pi}\frac{1-(-1)^n}{1+(1+n^2\pi^2)^2}.$$

Then

$$u(x,t) = \frac{4}{\pi}\sum_{m=1}^{\infty}\frac{(1+n^2\pi^2)\sin t - \cos t + e^{-(1+n^2\pi^2)t}}{(2m-1)[1+(1+(2m-1)^2\pi^2)^2]}\sin[(2m-1)\pi x].$$

4. A homogeneous rod of length $2L$ is compressed by forces applied on its ends so that it shrinks to a length $2L(1-\epsilon)$. At time $t=0$ the compression forces disappear. Find the subsequent longitudinal displacement of the rod for times $t>0$ provided the middle of the bar is located at $x=0$ and c is the longitudinal sound speed.

   Solution. The displacement of the rod is $u(x,0) = -\epsilon x$ ($[2L(1-\epsilon)-2L]/(2L) = -\epsilon$ is the uniform gradient $\partial u/\partial x$ before $t=0$, which gives $u(x,0) = -\epsilon x$). Solving the homogeneous wave equation with Neumann b.c. at $x = \pm L$ (the applied force at the rod ends is now zero), zero initial velocity and the initial profile $u(x,0) = -\epsilon x$ produces the displacement

$$u(x,t) = \frac{8\epsilon L}{\pi^2}\sum_{n=1}^{\infty}\frac{(-)^{n+1}}{(2n+1)^2}\sin\frac{(2n+1)\pi x}{2L}\,\frac{(2n+1)\pi ct}{2L},$$

   where c is the longitudinal sound velocity.

5. Solve the following Poisson equation inside the half-circle of radius 1 ($0<\theta<\pi$, $0\le r\le 1$):

$$\nabla^2 u = r\sin(2\theta),$$

   with the boundary conditions

$$u(1,\theta) = 0, \quad 0<\theta<\pi,$$
$$\frac{\partial u}{\partial\theta}(r,\theta) = 0, \quad \text{for } \theta = 0 \text{ and for } \theta = \pi.$$

   Solution. We try a particular solution $U = (Ar^3+Br^2)\sin(2\theta)$. Then $(6A+3A-4A)r+2B+B-4B = r$ which gives $A = 1/5$ and leaves B unspecified. Using $B=-A$, $U(1,\theta)=0$. Thus $v = u-U$ solves the Laplace equation with boundary conditions $v(1,\theta)=0$ and $\frac{\partial v}{\partial\theta} = 2r^2(r-1)/5$ for $\theta = 0,\pi$. Using separation of variables, $v(r,\theta) = a_0(r) + \sum_{n=1}^{\infty} a_n(r)\cos(n\theta)$. We now multiply the Laplace equation by $\cos(n\theta)$ and integrate by parts the result, thereby obtaining

$$\frac{\pi}{2}\left(a_n'' + \frac{1}{r}a_n' - \frac{n^2}{r^2}a_n\right) = -\frac{1}{r^2}\frac{\partial v}{\partial\theta}\cos(n\theta)\Big|_0^{\pi} = [1-(-1)^n]\frac{2}{5}(1-r),$$

   for $n\neq 0$ and $a_0'' + a_0'/r = 0$. From these equations, only the odd numbered modes are not zero. For odd n, we have

$$a_n'' + \frac{1}{r}a_n' - \frac{n^2}{r^2}a_n = \frac{8}{5\pi}(1-r), \quad a_n(1) = 0,\ |a_n(0)| < \infty.$$

We obtain

$$a_n(r) = -\frac{8(r^3 - r^n)}{5\pi(9 - n^2)} + \frac{8(r^2 - r^n)}{5\pi(4 - n^2)}, \quad n \neq 3,$$

and

$$a_3(r) = -\frac{8r^3 \ln r}{30\pi} - \frac{8(r^2 - r^3)}{25\pi}.$$

We have found the solution

$$\begin{aligned} u(r,\theta) &= \frac{r^3 - r^2}{5}\sin(2\theta) - \left(\frac{4r^3 \ln r}{15\pi} + \frac{8(r^2 - r^3)}{25\pi}\right)\cos(3\theta) \\ &+ \frac{8}{5\pi}\sum_{m=1, m\neq 2}^{\infty}\left[\frac{r^{2m-1} - r^3}{9 - (2m-1)^2} - \frac{r^{2m-1} - r^2}{4 - (2m-1)^2}\right]\cos[(2m-1)\theta]. \end{aligned}$$

A direct eigenfunction expansion $u(t,\theta) = \frac{1}{2}A_0(r) + \sum_{n=1}^{\infty} A_n(r)\cos(n\theta)$ *yields the following equation for* $a_n(r)$*:*

$$A_n'' + \frac{1}{r}A_n' - \frac{n^2}{r^2}A_n = \frac{2r}{\pi}\int_0^{\pi}\sin(2\theta)\cos(n\theta)d\theta = \frac{r}{\pi}\int_0^{\pi}\{\sin[(2+n)\theta] + \sin[(2-n)\theta]\}d\theta.$$

$$A_n'' + \frac{1}{r}A_n' - \frac{n^2}{r^2}A_n = \frac{r}{\pi}[1 - (-1)^n]\left(\frac{1}{2+n} + \frac{1}{2-n}\right) = \frac{4r}{\pi}\frac{1 - (-1)^n}{4 - n^2}.$$

The right hand side of this equation is 0 for $n = 2$*. Only the odd-numbered modes are not zero. For* $n \neq 3$ *there is a particular solution of the form* $b_n r^3$ *whereas for* $n = 3$ *the particular solution is* $b_3 r^3 \ln r$*. For a right hand side* $\mu_n r$*, we find* $b_n = \mu_n/(9 - n^2)$ *and* $b_3 = \frac{\mu_3}{6}$*. To these solutions we have to add solutions of the homogeneous equation so as to fulfill the boundary conditions* $|A_n(0)| < \infty$ *and* $A_n(1) = 0$*. The result is*

$$u(r,\theta) = -\frac{4}{15\pi}r^3 \ln r\cos(3\theta) + \frac{8}{\pi}\sum_{m=1, m\neq 2}^{\infty}\frac{r^3 - r^{2m-1}}{[4 - (2m-1)^2][9 - (2m-1)^2]}\cos[(2m-1)\theta].$$