EDOs & PDEs: final exam

Name: Group:

General Comments: Read carefully to answer all the questions. Since most of you are not very good at framing the final answer in a nice box, I have added a few in the exercises. You should do the computations in the blank papers and then include the final answer in the box provided. Please, USE THEM! Check the solutions whenever possible. If asked to explain something in your own words, **a few sentences** are enough. Separation of variables is a must! Not knowing how to do it might penalize you. Remember that you can assume the solution of $X'' + \lambda X = 0$ with D.B.C. and N.B.C. Good luck!

1. Consider the ODE

$$x dx + y(1 + 4x^2y^2 + 4y^4)dy = 0$$

(12) (a) Find an integrating factor of the form $\mu(x, y) = f(x^2 + y^2)$.

$\mu(x, y) =$

(6) (b) Using that integrating factor, find a potential that solves the ODE.

$V(x, y) =$

2. The **SIR** model is a simple mathematical model of epidemics where **S**, **I**, and **R** stand for **S**usceptible (people that are not infected but that are not immune), **I**nfected (people with the disease and that can transmit it to Susceptible people), and **R**ecovered (people who have recovered from the disease and are immune, so they can no longer be infected with the disease). Of course, more complicated models (for example including deceased, newborns or reinfected people) can be considered, but we will focus on this simplified model whose equations are:

$$\begin{cases} \mathbf{S}'(t) = -\beta \mathbf{S}(t)\mathbf{I}(t) \\ \mathbf{I}'(t) = \beta \mathbf{S}(t)\mathbf{I}(t) - \gamma \mathbf{I}(t) \\ \mathbf{R}'(t) = \gamma \mathbf{I}(t) \end{cases}$$

for some fixed parameters $\beta, \gamma > 0$.

(5) (a) Prove that $N(t) := \mathbf{S}(t) + \mathbf{I}(t) + \mathbf{R}(t)$ is constant. What does $N(t)$ represent?

(10) (b) Notice that $\mathbf{R}$ does not appear in the first two equations, so we have a (sub)system of two non-linear coupled ODEs:

$$\begin{cases} \mathbf{S}'(t) = -\beta \mathbf{S}(t)\mathbf{I}(t) \\ \mathbf{I}'(t) = \beta \mathbf{S}(t)\mathbf{I}(t) - \gamma \mathbf{I}(t) \end{cases}$$

Draw the $(\mathbf{S}, \mathbf{I})$ phase diagram (in the positive quadrant). Think before you start! It is much much simpler than usual and a couple of simple computations are needed.

(5) (c) What is the fate ($t \to \infty$) of the population with initial conditions $(\mathbf{S}_0, \mathbf{I}_0)$? Separate in cases if needed.

(4) (d) Explain in two sentences the meaning of the constants β and γ.

3. Given the potential $V(x) = \frac{x^6}{6} - \frac{5}{4}x^4 + 2x^2$

(3) (a) Write down the total energy for a particle of mass $m = 1$.

$E(t) =$

(5) (b) Assuming that the energy is conserved, write down a coupled system of equations for x and $y = \dot{x}$.

$\dot{x}(t) =$

$\dot{y}(t) =$

(10) (c) Sketch the $(x, \dot{x})$ phase portrait (make sure you add enough lines!).

(5) (d) Explain in one sentence what would happen if we add under-damping (adding to the ODE the term $-k\dot{x}$ with k small) AND sketch the resulting diagram.

(5) (e) Explain in one sentence what would happen if we add over-damping (adding to the ODE the term $-k\dot{x}$ with k large) AND sketch the resulting diagram.

(35) 4. Solve the following PDE:

$$\begin{aligned} &u_{tt} - u_{xx} + 2u_t + u = 0 \\ &u_x(t,0) = u_x(t,L) = 0 \\ &u(0,x) = f(x) \\ &u_t(0,x) = 0 \end{aligned}$$

$u(t,x) =$