ADVANCED MATHEMATICS

Bachelor Degree in Aerospace Engineering, Universidad Carlos III de Madrid

2021-2022

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Total duration: 3:00 hours

PROBLEM 1. (3 points) Calculate the solution $u(x, y, z)$ of Laplace's equation in the cube $[0, 1] \times [0, 1] \times [0, 1]$, using separation of variables:

$$\begin{cases} \Delta u = u_{xx} + u_{yy} + u_{zz} = 0, & 0 < x, y, z < 1, \\ u(0, y, z) = u(1, y, z) = 0, & 0 < y, z < 1, \\ u(x, 0, z) = u(x, 1, z) = 0, & 0 < x, z < 1, \\ \dfrac{d}{dz} u(x, y, 0) = 0, & 0 < x, y < 1, \\ u(x, y, 1) = f(x, y). \end{cases}$$

Find the solution if $f(x, y) = 1$.

Solution: If we write $u(x, y, z) = X(x)Y(y)Z(z)$, then the PDE becomes

$$X''YZ + XY''Z + XYZ'' = 0 \Rightarrow \frac{X''}{X} = -\frac{Y''}{Y} - \frac{Z''}{Z} = -\lambda.$$

So we obtain a first boundary value problem

$$\begin{cases} X''(x) + \lambda X(x) = 0, \\ X(0) = X(1) = 0. \end{cases}$$

This problem has eigenvalues $\lambda = \lambda_n = n^2\pi^2$, for $n = 1, 2, \ldots$, and eigenfunctions $X_n(x) = \sin(n\pi x)$. Then we separate the second equation:

$$-\frac{Y''}{Y} = \frac{Z''}{Z} - \lambda = \mu.$$

This gives a second boundary value problem:

$$\begin{cases} Y''(y) + \mu Y(y) = 0, \\ Y(0) = Y(1) = 0. \end{cases}$$

This problem has eigenvalues $\mu = \mu_m = m^2\pi^2$, for $m = 1, 2, \ldots$, and eigenfunctions $Y_m(y) = \sin(m\pi y)$. Finally, we have

$$Z''(z) = (\lambda_n + \mu_m)Z(z), \qquad Z'(0) = 0,$$

with solution $Z_{mn}(z) = \cosh(\sqrt{m^2 + n^2}\pi z)$. Appyling the superposition principle, we obtain

$$u(x, y, z) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn} \sin(n\pi x) \sin(m\pi y) \cosh(\sqrt{m^2 + n^2}\pi z).$$

Now we need to impose the last boundary condition:

$$f(x, y) = u(x, y, 1) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn} \sin(n\pi x) \sin(m\pi y) \cosh(\sqrt{m^2 + n^2}\pi).$$

We write this as

$$f(x,y) = \sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} a_{mn} \cosh(\sqrt{m^2+n^2}\pi) \sin(n\pi x) \right) \sin(m\pi y).$$

so we identify Fourier sine coefficients in the variable y:

$$\sum_{n=1}^{\infty} a_{mn} \cosh(\sqrt{m^2+n^2}\pi) \sin(n\pi x) = 2\int_0^1 f(x,y)\sin(m\pi y)dy,$$

and then we identify Fourier sine coefficients in the variable x:

$$\begin{aligned} a_{mn}\cosh(\sqrt{m^2+n^2}\pi) &= 2\int_0^1 \left(2\int_0^1 f(x,y)\sin(m\pi y)dy \right) \sin(n\pi x)dx, \\ &= 4\int_0^1 \int_0^1 f(x,y)\sin(m\pi y)\sin(n\pi x)dydx, \qquad n,m = 1,2,\ldots \end{aligned}$$

if $f(x,y)$ is continuous on $[0,1]\times[0,1]$.

If $f(x,y) = 1$, we obtain

$$\begin{aligned} a_{mn}\cosh\left(\sqrt{m^2+n^2}\pi\right) &= 4\int_0^1\int_0^1 \sin(m\pi y)\sin(n\pi x)dydx \\ &= 4\int_0^1 \sin(n\pi x)\int_0^1 \sin(m\pi y)dydx \\ &= \frac{4(1-(-1)^m)}{m\pi}\int_0^1 \sin(n\pi x)dx \\ &= \frac{4(1-(-1)^m)(1-(-1)^n}{mn\pi^2}. \end{aligned}$$

The solution is then

$$u(x,y,z) = \frac{4}{\pi^2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{(1-(-1)^m)(1-(-1)^n}{mn\cosh\left(\sqrt{m^2+n^2}\pi\right)} \sin(n\pi x)\sin(m\pi y)\cosh(\sqrt{m^2+n^2}\pi z).$$

PROBLEM 2. (0.75+0.75+0.5 points)

1. Given a second order linear ODE
$$w''(x) + a(x)w'(x) + b(x)w(x) = 0,$$
show that with a suitable factor $w(x) = \alpha(x)y(x)$ we can write the ODE without term involving the first derivative:
$$y''(x) + q(x)y(x) = 0.$$
Calculate $\alpha(x)$ and $q(x)$ in terms of the original coefficients $a(x)$ and $b(x)$.

2. Apply this transformation to Bessel's differential equation:
$$x^2w''(x) + xw'(x) + (x^2 - \nu^2)w(x) = 0.$$

3. Use the previous transformation to compute the general solution $w(x)$ of Bessel's differential equation if $\nu = \frac{1}{2}$.

Solution:

1. If we write $w(x) = \alpha(x)y(x)$, then
$$w'(x) = \alpha'(x)y(x) + \alpha(x)y'(x), \qquad w''(x) = \alpha''(x)y(x) + 2\alpha'(x)y'(x) + \alpha(x)y''(x),$$
and the ODE becomes
$$\alpha(x)y''(x) + \big(2\alpha'(x) + a(x)\alpha(x)\big)\, y'(x) + \big(\alpha''(x) + a(x)\alpha'(x) + \alpha(x)b(x)\big)\, y(x)) = 0$$
If we want the term that multiplies $y'(x)$ to be 0, we need
$$2\alpha'(x) + a(x)\alpha(x) = 0 \Rightarrow \alpha(x) = e^{-\frac{1}{2}\int^x a(t)dt},$$
and then
$$\begin{aligned}\alpha''(x) + a(x)\alpha'(x) + \alpha(x)b(x) &= \left(-\frac{1}{2}a'(x) + \frac{1}{4}a(x)^2 - \frac{1}{2}a(x)^2 + b(x)\right) e^{-\frac{1}{2}\int^x a(t)dt} \\ &= \left(-\frac{1}{2}a'(x) - \frac{1}{4}a(x)^2 + b(x)\right) e^{-\frac{1}{2}\int^x a(t)dt}\end{aligned}$$
Finally, we have
$$y''(x) + q(x)y(x) = 0, \qquad q(x) = b(x) - \frac{1}{2}a'(x) - \frac{1}{4}a(x)^2.$$

2. In this example, we have
$$a(x) = \frac{1}{x}, \qquad b(x) = 1 - \frac{\nu^2}{x^2},$$
so
$$q(x) = 1 - \frac{\nu^2}{x^2} + \frac{1}{2x^2} - \frac{1}{4x^2} = 1 - \frac{\nu^2}{x^2} + \frac{1}{4x^2} = 1 - \frac{\nu^2 - 1/4}{x^2}.$$

3. With $\nu = 1/2$, the transformed equation is
$$y''(x) + y(x) = 0,$$
with general solution $y(x) = c_1 \cos x + c_2 \sin x$. Since $a(x) = \frac{1}{x}$, we have
$$\alpha(x) = e^{-\frac{1}{2}\int^x \frac{dt}{t}} = \frac{1}{\sqrt{x}},$$
and the general solution of Bessel's differential equation with $\nu = \frac{1}{2}$ is
$$w(x) = c_1 \frac{\cos x}{\sqrt{x}} + c_2 \frac{\sin x}{\sqrt{x}}.$$

PROBLEM 3. (0.5+0.75+0.75 points) We consider the following ODE:

$$\phi''(x) + \lambda\phi(x) = 0, \qquad \phi'(0) = \phi(1) = 0.$$

1. Identify the coefficients in Sturm–Liouville form and determine if this is this a regular Sturm–Liouville problem.
2. Show that the eigenvalues are strictly positive, and use $u(x) = x^2 + ax + b$ as test function to obtain an estimation of the first eigenvalue.
3. Calculate the exact eigenvalues and eigenfunctions and check if the previous estimate is consistent.

Solution:

1. We identify the coefficients $p(x) = 1$, $q(x) = 0$ and $\sigma(x) = 1$, since they are all continuous and $p(x), \sigma(x) > 0$, we have a regular Sturm–Liouville problem on $[0, 1]$.

2. We use Rayleigh's quotient:

$$\lambda = \frac{-p(x)\phi(x)\phi'(x)\Big|_{x=0}^{x=1} + \int_0^1 \left[p(x)(\phi'(x))^2 - q(x)\phi(x)^2\right] dx}{\int_0^1 \phi(x)^2\sigma(x)dx} = \frac{0 + \int_0^1 (\phi'(x))^2 dx}{\int_0^1 \phi(x)^2 dx} \geq 0.$$

   Also, $\lambda = 0$ is not an eigenvalue, because in that case we obtain that the corresponding eigenfunction is 0.

   If we take the test function $u(x) = x^2 + ax + b$, the boundary conditions give $a = 0$ and $1 + a + b = 0$, so $u(x) = x^2 - 1$. Then

$$\lambda_1 \leq \frac{\int_0^1 (2x)^2 dx}{\int_0^1 (x^2-1)^2 dx} = \frac{\int_0^1 4x^2 dx}{\int_0^1 (x^4 - 2x^2 + 1)dx} = \frac{\frac{4}{3}}{\frac{1}{5} - \frac{2}{3} + 1} = \frac{\frac{4}{3}}{\frac{8}{15}} = \frac{5}{2} = 2.5$$

3. If we set $\phi(x) = e^{\alpha x}$, then we get $e^{\alpha x}(\alpha^2 + \lambda) = 0$. Since $e^{\alpha x} \neq 0$, we solve the quadratic equation, so $\alpha = \pm\sqrt{-\lambda}$. We know that $\lambda > 0$, so the solution is

$$\phi(x) = c_1 \cos\left(\sqrt{\lambda}x\right) + c_2 \sin\left(\sqrt{\lambda}x\right)$$

   If $\phi'(0) = 0$, we have $c_2 = 0$, and if $\phi(1) = 0$, we obtain the eigenvalues $\lambda_n = \left(\frac{\pi}{2} + n\pi\right)^2$, for $n = 0, 1, 2, \ldots$, with eigenfunctions

$$\phi(x) = \cos\left(\left(\frac{\pi}{2} + n\pi\right)x\right).$$

   The first eigenvalue is $\lambda_1 = \frac{\pi^2}{4} = 2.4674...$, so the previous estimate is consistent.

PROBLEM 4. (3 points) Given the following system:

$$\begin{cases} \dfrac{dx}{dt} = x(3 - x - y), \\ \dfrac{dy}{dt} = y(-2 + x), \end{cases}$$

compute and classify stationary points for $x, y \geq 0$ and draw the phase portrait.

Solution: We compute the stationary points of this system, which are $(0, 0)$, $(3, 0)$ and $(2, 1)$. Next, we linearise the problem near these points. The Jacobian of the system is

$$J(x, y) = \begin{pmatrix} 3 - 2x - y & -x \\ y & -2 + x \end{pmatrix}.$$

At the origin, we have

$$J(0, 0) = \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix},$$

with eigenvalues $\lambda = 3$ and $\lambda = -2$, so it is a saddle point. The eigenvectors are

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

At the stationary point $(3, 0)$, we have

$$J(3, 0) = \begin{pmatrix} -3 & -3 \\ 0 & 1 \end{pmatrix},$$

with eigenvalues $\lambda = -3$ and $\lambda = 1$, so it is another saddle point. The eigenvectors are

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} 3 \\ -4 \end{pmatrix}.$$

At the stationary point $(2, 1)$, we have

$$J(2, 1) = \begin{pmatrix} -2 & -2 \\ 1 & 0 \end{pmatrix}.$$

The eigenvalues satisfy $\lambda^2 + 2\lambda + 2 = 0$, with solutions $\lambda = -1 \pm i$, so it is a stable spiral point (because the real part of the eigenvalues is positive).

If we write local coordinates near this stationary point, $x(t) = 2 + \xi(t)$ and $y(t) = 1 + \eta(t)$, the first ODE is

$$\dot{\xi}(t) = -2\xi(t) - 2\eta(t).$$

If we pick a point on the positive vertical axis in the (ξ, η) plane, we have $\xi(t) = 0$ and $\eta(t) > 0$, so $\dot{\xi}(t) = -2\eta(t) < 0$, and the trajectories move counterclockwise.

The previous classification of stationary points corresponds to the linear system, but since none of them is a centre or improper node, they keep the same properties in the nonlinear system.

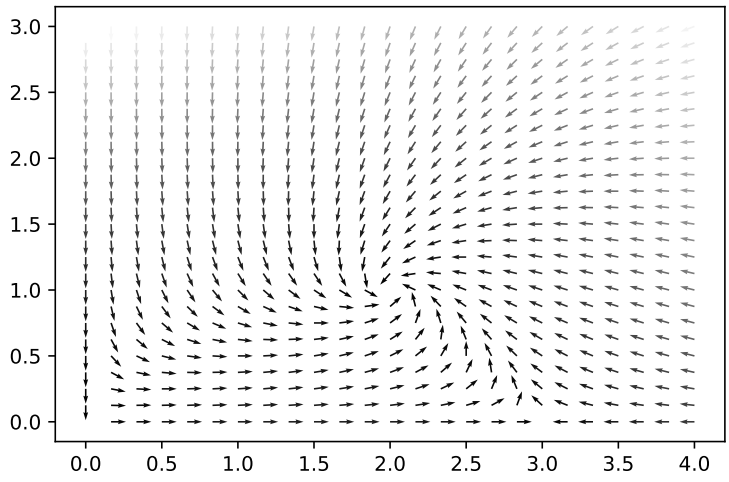


Figure 1: Phase portrait.