ADVANCED MATHEMATICS

Bachelor Degree in Aerospace Engineering, Universidad Carlos III de Madrid

2021-2022

June 20th, 2022

Total duration: 3:00 hours

In order to get full marks, you must explain (briefly) the results that you use from the course!

PROBLEM 1. (2 points) We consider the following initial value problem:

$$\frac{dx}{dt} = kx - x^2, \qquad y(0) = 1.$$

where $k \in \mathbb{R}$ is a constant.

1. Solve it as a separable ODE first, and then solve it again applying the change of variables $u(t) = \frac{1}{x(t)}$.
2. What is the behavior of the solution as $t \to \infty$?

PROBLEM 2. (3 points) Given the linear system of ODEs

$$\frac{d}{dt}\mathbf{x}(t) = M\mathbf{x}(t) \qquad M = \begin{pmatrix} 1 & -1 \\ 8 & -3 \end{pmatrix}, \qquad \mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$$

1. Find and classify any stationary point(s), sketch the phase portrait and compute the general solution.
2. Calculate the solution $\mathbf{u}(t)$ of the following inhomogeneous initial value problem:

$$\frac{d}{dt}\mathbf{u}(t) = M\mathbf{u}(t) + \begin{pmatrix} e^t \\ 4 \end{pmatrix}, \qquad \mathbf{u}(0) = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

PROBLEM 3. (3 points) We consider Laplace's equation (written in polar coordinates) in an half annulus Ω:

$$\begin{cases} \Delta u = u_{rr} + \dfrac{1}{r}u_r + \dfrac{1}{r^2}u_{\theta\theta} = 0, & 1 < r < 2, 0 < \theta < \pi, \\ u(r,0) = u(r,\pi) = 0, & 1 < r < 2, \\ u(1,\theta) = 0, & 0 < \theta < \pi, \\ u(2,\theta) = f(\theta), & 0 < \theta < \pi. \end{cases}$$

1. Using the divergence theorem, prove that the problem has a unique solution.
2. Solve the problem using separation of variables, with boundary value $f(\theta) = -\sin\theta + \sin 3\theta$.

PROBLEM 4. (2 points)

1. Calculate the general solution of the ODE

$$\ddot{x}(t) + \lambda\dot{x}(t) + \omega^2 x(t) = 0,$$

for $\lambda^2 < 4\omega^2$.

2. Determine λ such that the amplitude of the solution at time $t = 3$ is a hundredth of its initial value at $t = 0$.

3. Calculate the solution of the same ODE with $\lambda = 0$ and forcing term

$$\ddot{x}(t) + \omega^2 x(t) = a\cos\omega t.$$
