ADVANCED MATHEMATICS

Bachelor Degree in Aerospace Engineering, Universidad Carlos III de Madrid

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Total duration: 3:30 hours

PROBLEM 1. (2 points)

A chain of length $\ell = 5$m and mass m, with uniform mass density, is sitting on a table. If the chain is initially at rest and 1m of the chain hangs from the edge of the table, apply Newton's second law to calculate the time t_* when the whole chain falls off the table, neglecting any friction.

Hint: Formulate a second order ODE for the variable $x(t)$, which is the length of the portion of chain that is overhanging, as a function of time.

Solution: The total force acting on the chain is the weight of the overhanging part, which varies with time because the chain falls; since the mass density is uniform, the mass of this portion is $mg\frac{x(t)}{\ell}$. By Newton's second law, this is equal to mass times acceleration (of the whole chain), so $m\ddot{x}(t)$, and therefore the ODE is

$$\frac{mgx(t)}{\ell} = m\ddot{x}(t) \Rightarrow \ddot{x}(t) = \frac{g}{\ell}x(t).$$

This is a linear second order ODE with constant coefficients, so the general solution is

$$x(t) = Ae^{\sqrt{\frac{g}{\ell}}t} + Be^{-\sqrt{\frac{g}{\ell}}t}.$$

The initial conditions are $x(0) = 1$ and $x'(0) = 0$ (one meter of chain hanging and zero velocity), so $A = B$ and $A + B = 1$, therefore $A = B = \frac{1}{2}$, and

$$x(t) = \cosh\left(\sqrt{\frac{g}{\ell}}t\right).$$

We want to find t_* such that $x(t_*) = \ell = 5$, so

$$t_* = \frac{\sqrt{5}}{\sqrt{9.8}}\cosh^{-1}(5) = 1.6374...\text{s}.$$

PROBLEM 2. (3 points)

Consider a greenhouse/rockbed system. During the day, air in the greenhouse is heated primarily by solar radiation. Whenever the air temperature in the greenhouse exceeds an upper threshold value, a thermostatically controlled fan circulates the air through the system, thereby transferring heat to the rockbed. At night, when the air temperature in the greenhouse drops below a lower threshold value, the fan again turns on, and heat stored in the rockbed warms the circulating air. We wish to study temperature variation in the greenhouse during the nighttime phase of the cycle. A simplified model for the system denotes the air temperature in the greenhouse by $u_1(t)$ and the temperature of the rockbed by $u_2(t)$. The vector $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ is called the state vector of the system. We will measure t in hours and temperature in degrees Celsius. The pair of differential equations with two unknown temperature functions u_1 and u_2 is

$$\begin{cases} \dfrac{du_1}{dt} = -\dfrac{13}{8}u_1 + \dfrac{3}{4}u_2 + 14, \\ \dfrac{du_2}{dt} = \dfrac{1}{4}u_1 - \dfrac{1}{4}u_2, \end{cases}$$

or

$$\mathbf{u}' = A\mathbf{u} + \mathbf{b},\ A = \begin{pmatrix} -\frac{13}{8} & \frac{3}{4} \\ \frac{1}{4} & -\frac{1}{4} \end{pmatrix},\ \mathbf{b} = \begin{pmatrix} 14 \\ 0 \end{pmatrix},\ \mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix},$$

together with the initial conditions

$$u_1(0) = 0, \quad u_2(0) = 36,$$

constitute a mathematical model for the variation of air temperature in the greenhouse and the temperature of the rockbed during the nighttime phase of the cycle. Solutions of these equations for $u_1(t)$ and $u_2(t)$ can assist us in designing a solar-powered rockbed heating system for the greenhouse.

2.1 Find the general solution $\mathbf{u}_h$ of the homogeneous system.

2.2
- Find the critical point $\mathbf{u}_0$ of the non-homogeneous system.
- Now using such an equilibrium point perform a shift to the origin of the phase plane to the critical point applying the coordinate transformation $\mathbf{u} = \mathbf{u}_0 + \mathbf{x}$. Find the solution $\mathbf{x}$ after performing such a shift for the particular initial conditions.
- Then, find the general solution of the non-homogenous problem.
- Plot each component u_1 and u_2 versus t assuming the initial conditions. What can you conclude about the behaviour of each one of those functions with respect to each other? Describe how the behaviour of both components in time (their evolution).

2.3 Describe the behaviour of the trajectories $\mathbf{u} = [u_1, u_2]^T$ in a neighbourhood of the critical point for the general solutions.

Solution:

2.1 To solve the homogeneous system, we let $\mathbf{u} = e^{\lambda t}\mathbf{v}$, arriving at the eigenvalue problem

$$(A - \lambda\mathbb{I})\mathbf{v} = \begin{pmatrix} -\frac{13}{8} - \lambda & \frac{3}{4} \\ \frac{1}{4} & -\frac{1}{4} - \lambda \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

for the matrix A of coefficients in system. Thus the characteristic equation is

$$\lambda^2 + \frac{15}{8}\lambda + \frac{7}{32} = 0,$$

so the eigenvalues are $\lambda_1 = -\frac{7}{4}$ and $\lambda_2 = -\frac{1}{8}$. If $\lambda = \lambda_1 = -\frac{7}{4}$ we get to the system

$$\begin{pmatrix} \frac{1}{8} & \frac{3}{4} \\ \frac{1}{4} & \frac{3}{2} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

Thus, we obtain that $\frac{1}{4}v_1 + \frac{3}{2}v_2 = 0$, so we take the eigenvector $\mathbf{v}_1$ associated with the eigenvalue $\lambda_1 = -\frac{7}{4}$

$$\mathbf{v}_1 = \begin{pmatrix} 6 \\ -1 \end{pmatrix}.$$

Similarly, if $\lambda = \lambda_2 = -\frac{1}{8}$ it follows that

$$\begin{pmatrix} -\frac{3}{2} & \frac{3}{4} \\ \frac{1}{4} & -\frac{1}{8} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Hence, $2v_1 - v_2 = 0$, and the associated eigenvector will be

$$\mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

The corresponding solutions of the homogeneous system are

$$\mathbf{u}_1(t) = e^{-7/4t}\begin{pmatrix} 6 \\ -1 \end{pmatrix}, \quad \mathbf{u}_2(t) = e^{-1/8t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Moreover, the Wronskian of these solutions is

$$W = \det\begin{pmatrix} 6e^{-7/4t} & e^{-1/8t} \\ -e^{-7/4t} & 2e^{-1/8t} \end{pmatrix} = 13e^{-15/8t} \neq 0,$$

which is never zero. Hence the solutions $\mathbf{u}_1$ and $\mathbf{u}_2$ form a fundamental set, and the general solution of the homogeneous system is

$$\mathbf{u}(t) = c_1\mathbf{u}_1(t) + c_2\mathbf{u}_2(t) = c_1e^{-7/4t}\begin{pmatrix} 6 \\ -1 \end{pmatrix} + c_2e^{-1/8t}\begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

with c_1 and c_2 as two arbitrary constants.

2.2 To find the critical point, we set $\frac{du_1}{dt} = 0$ and $\frac{du_2}{dt} = 0$ such that

$$\begin{aligned} -\frac{13}{8}u_1 + \frac{3}{4}u_2 + 14 = 0, \\ \frac{1}{4}u_1 - \frac{1}{4}u_2 = 0. \end{aligned}$$

Thus the only critical point of the system is $u_1 = u_2 = 16$, i.e. $\mathbf{u}_0 = [16, 16]^T$.

Now, let $x_1(t)$ and $x_2(t)$ be the deviations of $u_1(t)$ and $u_2(t)$ from their respective equilibrium values, that is,

$$x_1(t) = u_1(t) - 16, \quad x_2(t) = u_2(t) - 16.$$

Thus,

$$u_1(t) = x_1(t) + 16, \quad u_2(t) = x_2(t) + 16,$$

such that, after substituting them into the non-homogeneous system the pair $\mathbf{x} = [x_1(t)\ x_2(t)]^T$ solves the system

$$\begin{cases} \dfrac{dx_1}{dt} = -\dfrac{13}{8}x_1 + \dfrac{3}{4}x_2, \\ \dfrac{dx_2}{dt} = \dfrac{1}{4}x_1 - \dfrac{1}{4}x_2, \end{cases}$$

whose general solution is

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) = c_1e^{-7/4t}\begin{pmatrix} 6 \\ -1 \end{pmatrix} + c_2e^{-1/8t}\begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

with c_1 and c_2 two arbitrary constants. However, after such a shifting the initial conditions become

$$x_1(0) = -16, \quad x_2(0) = 20.$$

Therefore, the constants are fixed by the values

$$c_1 = -4, \quad c_2 = 8,$$

so that

$$\mathbf{x}(t) = -4e^{-7/4t}\begin{pmatrix} 6 \\ -1 \end{pmatrix} + 8e^{-1/8t}\begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

and, hence, the general solution of the non-homogenous problem is

$$\mathbf{u}(t) = \mathbf{x}(t) + \mathbf{u}_0 = -4e^{-7/4t}\begin{pmatrix} 6 \\ -1 \end{pmatrix} + 8e^{-1/8t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 16 \\ 16 \end{pmatrix}.$$

The variables x_1 and x_2 specify the temperatures of the air and the rockbed relative to their respective equilibrium values, whereas u_1 and u_2 are the corresponding actual temperatures. The result of using x_1 and x_2 instead of u_1 and $u_2$is to simplify the system by eliminating the non-homogeneous term. In geometrical language we have shifted coordinates so that the critical point $[16, 16]^T$ in the u_1u_2-plane is now located at the origin in the x_1x_2-plane.

We see that the temperature u_1 of the air, ostensibly heated by the rockpile, rises rapidly, in 2.3h, from 0C to approximately 21.6C. During that same period, the rockpile cools from 36C to roughly 28C, a change of only 8C. Indeed, both u_1 and u_2 slowly approach the same constant limiting value as $t \to \infty$. From the general solution we see that

$$\lim_{t\to\infty} \mathbf{u}(t) = \begin{pmatrix} 16 \\ 16 \end{pmatrix},$$

since all other terms in the solution contain one of the exponential factors, $e^{-t/8}$ or $e^{-7t/4}$, which tend to 0, as $t \to \infty$. Note that the constant vector $\mathbf{u} = 16\mathbf{i} + 16\mathbf{j}$ is also a solution. This solution satisfies the initial condition

$$\mathbf{u}(0) = \begin{pmatrix} 16 \\ 16 \end{pmatrix}.$$

Physically, this solution corresponds to thermal equilibrium: the temperature of the greenhouse air and the temperature of the rockbed, if set initially to the outside air temperature 16C, do not change in time.

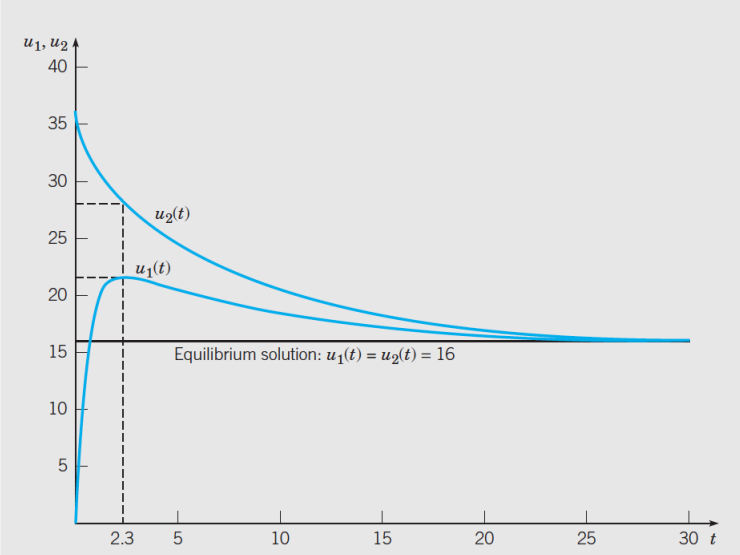


Figure 1: Graphs of component plots u_1 and u_2.

2.3 As t advances, the tip of the vector $\mathbf{u}$ traces a curve in the phase plane, called a trajectory or orbit, that graphically displays the path of the state of the system in the state space. Each trajectory in the phase portrait is generated by plotting, in the u_1u_2-plane, the set of points with coordinates $(u_1(t), u_2(t))$ for several values of t, and then drawing a curve through the resulting points. The values of $u_1(t)$ and $u_2(t)$ have been obtained through the analytical solutions shown above.

To visualize the solution, it is helpful to consider trajectories in the x_1x_2-plane

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) = c_1e^{-7/4t}\begin{pmatrix} 6 \\ -1 \end{pmatrix} + c_2e^{-1/8t}\begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

for various values of the constants c_1 and c_2 and shifting it to the critical point $[16, 16]^T$. Thus, if $c_2 = 0$ we get $\mathbf{x} = c_1\mathbf{x}_1(t)$ or, in scalar form,

$$x_1 = 6c_1e^{-7t/4}, \quad x_2 = -c_1e^{-7t/4},$$

then, by eliminating t between these two equations, we see that this solution lies on the straight line $x_2 = -\frac{x_1}{6}$. This is the line through the origin in the direction of the eigenvector $\mathbf{v}_1$. If we look on the solution as the trajectory of a moving particle, then the particle is in the fourth quadrant when $c_1 > 0$ and in the second quadrant when $c_1 < 0$. In either case, the particle moves toward the origin as t increases since the eigenvalue is negative. Next we set $c_1 = 0$ to get $\mathbf{x} = c_2\mathbf{x}_2(t)$, or

$$x_1 = c_2 e^{-t/8}, \quad x_2 = 2c_2 e^{-t/8},$$

then, the solution lies on the line $x_2 = 2x_1$, whose direction is determined by the eigenvector $\mathbf{v}_2$. This solution is in the first quadrant when $c_2 > 0$ and in the third quadrant when $c_2 < 0$. In both cases the particle moves toward the origin as t increases.

The solution is a combination of $x_1(t)$ and $x_2(t)$, so all solutions approach the origin as $t \to \infty$. For large t the term $c_2x_2(t)$ is dominant and the term $c_1x_1(t)$ is negligible in comparison. Thus all solutions for which $c_2 = 0$ approach the origin tangent to the line $x_2 = 2x_1$ as $t \to \infty$.

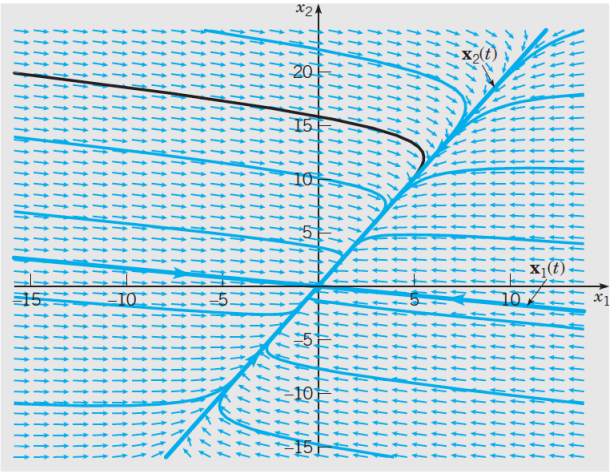


Figure 2: x_1x_2-Phase portrait.

Observe that the phase portrait for the original problem is essentially the same as the x_1x_2-phase plane, except that now the critical point occurs at $[16, 16]^T$ instead of at the origin.

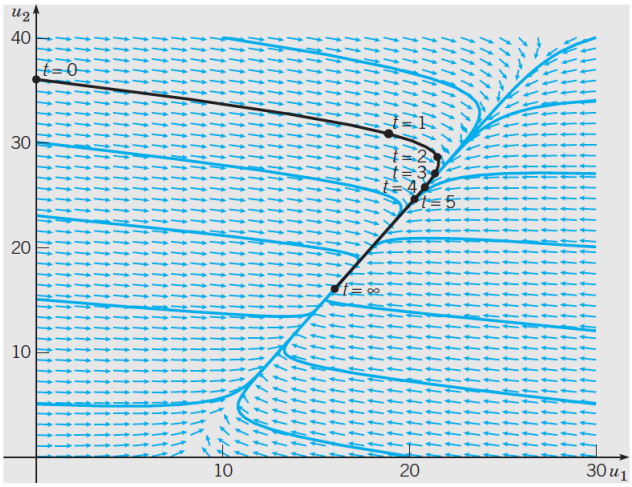


Figure 3: u_1u_2-Phase portrait.

PROBLEM 3. (3 points) Solve the following PDE:

$$\begin{cases} u_{tt} - u_{xx} + 2u_t + u = 0, & 0 < x < 1, t > 0, \\ u(0,t) = u(1,t) = 0, & t > 0, \\ u(x,0) = f(x), & 0 < x < 1, \\ u_t(x,0) = 0, & 0 < x < 1. \end{cases}$$

Find the solution if $f(x) = x(1-x)$.

Solution: We apply separation of variables, so $u(x,t) = X(x)T(t)$:

$$XT'' - X''T + 2XT' + XT = 0 \Rightarrow \frac{T'' + 2T' + T}{T} = \frac{X''}{X} = -\lambda,$$

where λ is a constant. We begin with the ODE for $X(x)$:

- If $\lambda < 0$ then
$$X(x) = Ae^{\sqrt{-\lambda}x} + Be^{\sqrt{-\lambda}x},$$
and the boundary conditions $X(0) = X(1) = 0$ give $A = -B$. Hence, $A\sinh(\sqrt{-\lambda}) = 0$, so $A = B = 0$, the trivial solution.

- If $\lambda = 0$, then
$$X(x) = Ax + B,$$
and the boundary conditions $X(0) = X(1) = 0$ give $A = B = 0$, the trivial solution again.

- If $\lambda > 0$, then
$$X(x) = A\cos\left(\sqrt{\lambda}x\right) + B\sin\left(\sqrt{\lambda}x\right),$$
and now the boundary conditions $X(0) = X(1) = 0$ give $A = 0$ and $B\sin(\sqrt{\lambda}) = 0$. Therefore, either $B = 0$ (trivial solution once again) or we get the eigenvalues $\lambda = n^2\pi^2$ and eigenfunctions $X_n(x) = \sin(n\pi x)$, with $n \in \mathbb{N}$.

On the other hand, the ODE for $T(t)$ is

$$T'' + 2T' + (1+\lambda)T = T'' + 2T' + \left(1 + n^2\pi^2\right)T = 0.$$

Since it is a linear second order ODE with constant coefficients, we set $T(t) = e^{rt}$. Then we get

$$e^{rt}(r^2 + 2r + (1+\lambda)) = 0.$$

Since $e^{rt} \neq 0$, we solve the quadratic equation, which has solutions

$$r_\pm = \frac{-2 \pm \sqrt{4 - 4(1+\lambda)}}{2} = -1 \pm n\pi i,$$

and therefore

$$T_n(t) = Ae^{-t}\cos(n\pi t) + Be^{-t}\sin(n\pi t), \quad \text{with} \quad n \in \mathbb{N}.$$

It follows that

$$T_n'(t) = Ae^{-t}\left(-\cos(n\pi t) - n\pi\sin(n\pi t)\right) + Be^{-t}\left(-\sin(n\pi t) + n\pi\cos(n\pi t)\right),$$

and then if $T_n'(0) = 0$, we have $-A + n\pi B = 0$, so

$$T_n(t) = Ae^{-t}\left[\cos(n\pi t) + \frac{1}{n\pi}\sin(n\pi t)\right],$$

and by the superposition principle, we have the solution

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-t} \left[\cos(n\pi t) + \frac{1}{n\pi}\sin(n\pi t)\right] \sin(n\pi x).$$

Furthermore, $u(x,0) = f(x)$, so

$$f(x) = \sum_{n=1}^{\infty} a_n \sin(n\pi x),$$

so a_n are the Fourier sine coefficients of the function $f(x)$:

$$a_n = 2\int_0^1 f(x)\sin(n\pi x)\,dx, \qquad n \geq 1.$$

If $f(x) = x(1-x)$, then

$$a_n = 2\int_0^1 x(1-x)\sin(n\pi x)\,dx = 2\int_0^1 x\sin(n\pi x)\,dx - 2\int_0^1 x^2\sin(n\pi x)\,dx.$$

Since

$$\begin{aligned}
\int_0^1 x\sin(n\pi x)\,dx &= -\frac{x}{n\pi}\cos(n\pi x)\Big|_{x=0}^{x=1} + \frac{1}{n\pi}\int_0^1 \cos(n\pi x)\,dx = \frac{(-1)^{n+1}}{n\pi},\\
\int_0^1 x^2\sin(n\pi x)\,dx &= -\frac{x^2}{n\pi}\cos(n\pi x)\Big|_{x=0}^{x=1} + \frac{2}{n\pi}\int_0^1 x\cos(n\pi x)\,dx\\
&= \frac{(-1)^{n+1}}{n\pi} + \frac{2}{n\pi}\left[\frac{x}{n\pi}\sin(n\pi x)\Big|_{x=0}^{x=1} - \frac{1}{n\pi}\int_0^1 \sin(n\pi x)\,dx\right]\\
&= \frac{(-1)^{n+1}}{n\pi} + \frac{2}{n^3\pi^3}\cos(n\pi x)\Big|_{x=0}^{x=1}\\
&= \frac{(-1)^{n+1}}{n\pi} - \frac{2(1-(-1)^n)}{n^3\pi^3},
\end{aligned}$$

then

$$a_n = \frac{4(1-(-1)^n)}{n^3\pi^3},$$

and finally

$$\begin{aligned}
u(x,t) &= \sum_{n=1}^{\infty} \frac{4(1-(-1)^n)}{n^3\pi^3} e^{-t}\left[\cos(n\pi t) + \frac{1}{n\pi}\sin(n\pi t)\right]\sin(n\pi x)\\
&= \frac{8}{\pi^3}\sum_{n=1}^{\infty} \frac{e^{-t}}{(2n-1)^3}\left[\cos((2n-1)\pi t) + \frac{1}{(2n-1)\pi}\sin((2n-1)\pi t)\right]\sin((2n-1)\pi x).
\end{aligned}$$

PROBLEM 4. (2 points) Consider the following eigenvalue problem on $[1,3]$:

$$\begin{cases} x^2\phi''(x) - x\phi'(x) - x\phi(x) + \lambda x^2\phi(x) = 0,\\ \phi(1) = \phi(3) = 0. \end{cases}$$

4.1 Is it a Sturm-Liouville problem? If not, convert it into Sturm-Liouville form, identifying $p(x)$, $q(x)$, and $\sigma(x)$.

4.2 Use the Rayleigh quotient to show if the eigenvalues λ are positive.

4.3 What can we say about the eigenvalues and eigenfunctions? What orthogonality property do we have for the eigenfunctions?

Solution:

4.1 If this were a Sturm-Liouville problem, $x^2\phi''(x) - x\phi'(x)$ should be equal to $(p(x)\phi')'$ for some function $p(x)$. Therefore, we should have that $p(x) = x^2$ and $p'(x) = -x$. As this is impossible, the eigenvalue problem is not a Sturm-Liouville problem.

In order to bring it into Sturm-Liouville form, we multiply the equation by the integrating factor $\mu(x)$, which is chosen so that $\mu(x)x^2\phi''(x) - \mu(x)x\phi'(x) = (p(x)\phi')'$. Then $p(x) = \mu(x)x^2$, $p'(x) = -x\mu(x)$, which gives $\mu(x) = x^{-3}$ and $p(x) = x^{-1}$. The resulting equation in Sturm-Liouville form is

$$\frac{d}{dx}\left(\frac{1}{x}\phi'\right) - \frac{1}{x^2}\phi + \frac{\lambda}{x}\phi = 0.$$

4.2 The Rayleigh quotient can be easily obtained from an equation in Sturm-Liouville form. We multiply each term of the equation by ϕ and integrate by parts, i.e.

$$\lambda = \frac{-\frac{1}{x}\phi\phi']_1^3 + \int_1^3 \frac{1}{x}(\phi')^2\,dx + \int_1^3 \frac{1}{x^2}\phi^2\,dx}{\int_1^3 \frac{1}{x}\phi^2\,dx} \geq 0,$$

where the first term in the numerator vanishes due to the boundary conditions $\phi(1) = \phi(3) = 0$. We conclude that λ is non-negative, as so are the integrands in the numerator and the denominator. We now know that $\lambda \geq 0$, but we still have to check whether $\lambda = 0$ can be an eigenvalue. For $\lambda = 0$, we should have

$$\int_1^3 \frac{1}{x}(\phi')^2\,dx + \int_1^3 \frac{1}{x^2}\phi^2\,dx = 0.$$

However, both terms are non-negative so that it is actually sufficient to have

$$\int_1^3 \frac{1}{x^2}\phi^2\,dx = 0,$$

since from the first term we just conclude that the eigenfunction must be constant. Thus, thanks to the final condition and since the eigenfucntion must be continuous, we just arrive at the trivial solution $\phi = 0$. Then, $\lambda = 0$ cannot be an eigenvalue. Therefore, $\lambda > 0$.

4.3 Thanks to the Sturm-Liouville Theorem, despite of not having computed the eigenvalues for our problem explicitly. Just having an estimation for the first eigenvalue by using a test function, we can say that:

- There are infinitely many eigenvalues, all of them real, which can be arranged in an increasing, unbounded sequence, $\lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots$.
- Eigenfunctions corresponding to different eigenvalues, which can be chosen real as well, are orthogonal with respect to the inner product

$$\int_1^3 \frac{1}{x}\phi_n(x)\phi_m(x)\,dx.$$