ADVANCED MATHEMATICS

Bachelor Degree in Aerospace Engineering, Universidad Carlos III de Madrid

January 14th, 2020

Total duration: 3.5 hours

PROBLEM 1. (2.5 points)

Consider the following non-homogeneous ordinary differential equation:

$$x^2\, y'' - 2\, x\, y' + 2\, y = x\, \log x.$$

1.1 Find the general solution of the associated homogeneous equation.

1.2 Find the general solution of the non-homogeneous equation.

1.3 Find the unique solution of the boundary value problem given by the non-homogeneous equation and the boundary conditions $y(1) = 0$ and $y(2) = -(2 + \log 2) \log 2$.

PROBLEM 2. (2.5 points)

Consider the system $\begin{cases} x' = \alpha x + y, \\ y' = -x + \alpha y + (\alpha^2 + 1)x^3, \end{cases}$ in which $\alpha \in \mathbb{R}$ is a parameter.

2.1 Analyze and classify the fixed points in terms of the values of α.

2.2 Sketch the phase portrait for $\alpha = 0$.

2.3 For $\alpha = 0$, find y as a function of x thus describing the family of all integral curves of the system.

PROBLEM 3. (2.5 points)

Consider the following problem for a vibrating string on an interval, which is described by a modified wave equation in which $\gamma > 0$ is a parameter:

$$\begin{cases} u_{tt} = u_{xx} - \gamma u, & 0 < x < \pi,\ t > 0, \\ u(0,t) = 0, \ \ u_x(\pi, t) = 0, & t > 0, \\ u(x,0) = \sin(3x/2), & 0 < x < \pi, \\ u_t(x,0) = 0, & 0 < x < \pi. \end{cases}$$

3.1 Interpret physically the boundary conditions.

3.2 Solve the problem.

3.3 Based on the structure of the solution, what kind of physical effect does the term proportional to γ have in this differential equation, as compared to the standard wave equation?

PROBLEM 4. (2.5 points)

Consider the following non-homogeneous heat-conduction problem, where a time-independent source occurs:

$$\begin{cases} u_t = u_{xx} + 12x^2 - 6x, & 0 < x < 1,\ t > 0, \\ u(0,t) = 2, \ \ u(1,t) = 3, & t > 0, \\ u(x,0) = f(x), & 0 < x < 1, \end{cases}$$

4.1 Find a steady-state solution if it exists.

4.2 Find the solution of the full non-homogeneous problem.

4.3 Analyze the $t \to \infty$ limit of the solution found in 4.2, justifying your answer.
