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Enero 2019 con solución

2° Ampliación de Matemáticas

Grado en Ingeniería Aeroespacial

Escuela Politécnica Superior
Universidad Carlos III de Madrid

Como aún estás en la portada, es momento de redes sociales. Cotilléanos y luego a estudiar.

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ADVANCED MATHEMATICS

Bachelor Degree in Aerospace Engineering, Universidad Carlos III de Madrid

Course 2018-2019

January 21, 2019

Total duration: 4 hours

PROBLEM 1. (2.5 points) Consider the following first order linear equation

$$xy'' + (1-2x)y' + (x-1)y = (x+1)e^x\,, \qquad x > 0\,.$$

(i) Using that $y_1 = e^x$ is a solution to the corresponding homogeneous equation, find a second solution (of the type $y_2 = vy_1$) and thus obtain the general solution of the corresponding homogeneous linear equation.

(ii) Find the general solution of the complete equation.

Solution:

1. (i) A second solution of the form $y_2 = vy_1$ for an equation $y'' + P(x)y' + Q(x)y = 0$, must satisfy

$$v' = \frac{1}{y_1^2}\, e^{-\int P\,dx}$$

and, hence, in our particular case

$$v' = \frac{1}{e^{2x}}\, e^{-\int \frac{1-2x}{x}\,dx} = \frac{1}{e^{2x}}\, e^{-\log x} e^{2x} = \frac{1}{x}\,.$$

Then, $v = \log x$, such that $y_2 = e^x \log x$. Therefore, the general solution for the homogeneous equation will be

$$y_{GH} = c_1 e^x + c_2 e^x \log x\,.$$

(ii) Applying the method of variation of constants we will look for a second solution of the form $y_p = v_1(x)\, e^x + v_2(x)\, e^x \log x$. Thanks to this method v_1', v_2' satisfy

$$\begin{cases} v_1'\, e^x + v_2'\, e^x \log x = 0\,, \\ v_1'\, e^x + v_2'\, e^x (\dfrac{1}{x} + \log x) = \dfrac{x+1}{x} e^x\,, \end{cases} \iff \begin{cases} v_1' + v_2' \log x = 0\,, \\ v_1' + v_2' (\dfrac{1}{x} + \log x) = \dfrac{x+1}{x}\,. \end{cases}$$

Using Cramer's method,

$$v_1' = \frac{\begin{vmatrix} 0 & \log x \\ \frac{x+1}{x} & \frac{1}{x} + \log x \end{vmatrix}}{\begin{vmatrix} 1 & \log x \\ 1 & \frac{1}{x} + \log x \end{vmatrix}} = \frac{-\frac{x+1}{x} \log x}{\frac{1}{x}} = -(x+1)\, \log x\,,$$

$$v_2' = \frac{\begin{vmatrix} 1 & 0 \\ 1 & \frac{x+1}{x} \end{vmatrix}}{\begin{vmatrix} 1 & \log x \\ 1 & \frac{1}{x} + \log x \end{vmatrix}} = \frac{\frac{x+1}{x}}{\frac{1}{x}} = x+1\,.$$

Consequently, integrating by parts,

$$v_1 = -\int (x+1)\log x\,dx = -\left(\frac{x^2}{2} + x\right)\log x + \frac{x^2}{4} + x\,, \qquad v_2 = \frac{x^2}{2} + x\,,$$

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it follows that

$$y_p = v_1\, e^x + v_2\, e^x \log x = \left(\frac{x^2}{4} + x\right)\, e^x\,.$$

Finally, the general solution for the non-homogeneous equation is

$$y = c_1 e^x + c_2 e^x \log x + \left(\frac{x^2}{4} + x\right)\, e^x\,.$$

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PROBLEM 2

$$\left.\begin{aligned}\dot{x} &= x(4-x-y)\\ \dot{y} &= y(2+2\alpha-y-\alpha x)\end{aligned}\right\} \qquad \begin{aligned}&\alpha > 0\\ &x(t), y(t) \geq 0\end{aligned}$$

2.1) The equation for x is logistic, with an additional "sink"-like term for x that decreases its growth rate once the y species is present ($y \neq 0$). Except for numerical values, the same applies to y with respect to x. Hence, the two species are competing with each other beyond being subject to logistic-type dynamics.

2.2)
$$\left.\begin{aligned}x(4-x-y)&=0\\ y(2+2\alpha-y-\alpha x)&=0\end{aligned}\right\} \longrightarrow \begin{cases} x=0 \rightarrow y(2+2\alpha-y)=0 \begin{cases} y=0 \\ y=2+2\alpha \end{cases} \\ x+y=4 \rightarrow \begin{cases} y=0 \\ y+\alpha x=2+2\alpha \end{cases}\end{cases}$$

$$y+4\alpha-\alpha y=2+2\alpha \rightarrow \begin{cases} y=\dfrac{2-2\alpha}{1-\alpha}=2 \\ x=2 \end{cases}$$

Hence, starting from the first equation we get 3 critical points

$$P_1=(0,0),\ P_2=(0,2+2\alpha),\ P_3=(2,2)$$

Starting from the second we find another one,

$$y=0 \begin{cases} x=0 \ (P_1) \\ 4-x=0 \rightarrow x=4 \end{cases} \qquad P_4=(4,0)$$

The Jacobian matrix is

$$J(x,y)=\begin{pmatrix} 4-2x-y & -x \\ -\alpha y & 2+2\alpha-2y-\alpha x \end{pmatrix}$$

Hence:

$$J(0,0)=\begin{pmatrix} 4 & 0 \\ 0 & 2+2\alpha \end{pmatrix} \rightarrow \left.\begin{aligned}\lambda_1 &= 4>0\\ \lambda_2 &= 2+2\alpha > 0 \ \forall\alpha\end{aligned}\right\}$$

($\alpha > 0$)

$P_1\ (0,0)$ is an unstable node $\forall\alpha$.

$$J(0,2+2\alpha)=\begin{pmatrix} 4-2-2\alpha & 0 \\ -\alpha(2+2\alpha) & 2+2\alpha-4-4\alpha \end{pmatrix}=\begin{pmatrix} 2-2\alpha & 0 \\ -2\alpha-2\alpha^2 & -2-2\alpha \end{pmatrix}$$

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$\rightsquigarrow \lambda_1 = 2(1-\alpha)$
$\lambda_2 = -2-2\alpha < 0 \ \forall \alpha$

$P_2(0, 2+2\alpha)$ is a { saddle point $0<\alpha<1$; stable node $\alpha>1$ }

$$J(2,2) = \begin{pmatrix} 4-4-2 & -2 \\ -2\alpha & 2+2\alpha-4-2\alpha \end{pmatrix} = \begin{pmatrix} -2 & -2 \\ -2\alpha & -2 \end{pmatrix}$$

Eigenvalues: $\begin{vmatrix} -2-\lambda & -2 \\ -2\alpha & -2-\lambda \end{vmatrix} = 0 = \lambda^2 + 4\lambda + 4 - 4\alpha$

$$\lambda = \frac{-4 \pm \sqrt{16-16+16\alpha}}{2} = -2 \pm 2\alpha^{1/2}$$

$P_3(2,2)$ is a { stable node, $\alpha<1$; saddle, $\alpha>1$ }
interior point

$\alpha_0 = 1$

{ $2(\alpha-1)$; $-2-2\alpha<0$ }

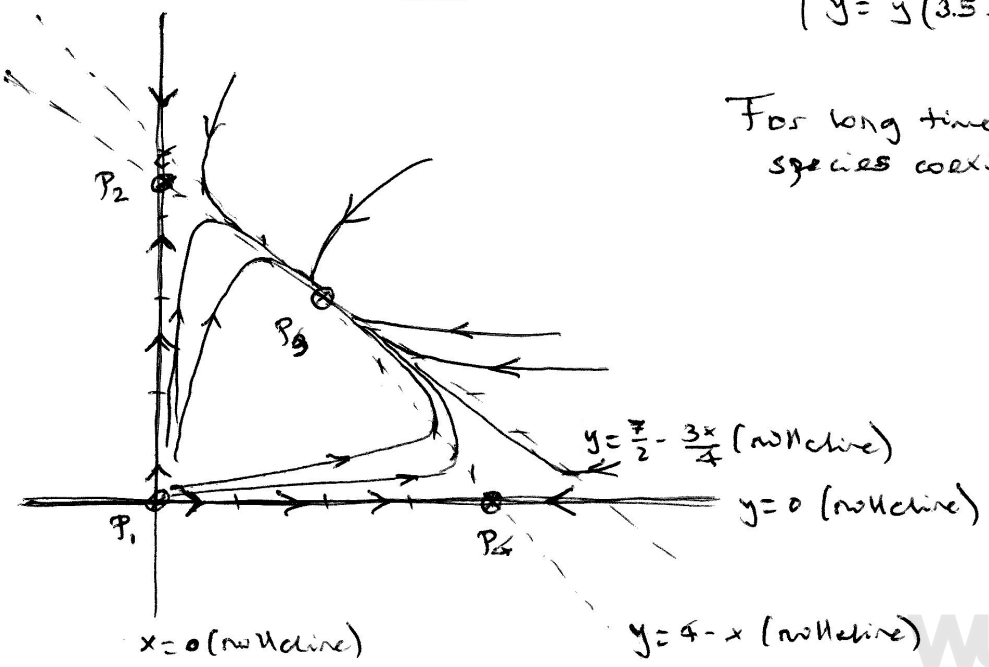
$$J(4,0) = \begin{pmatrix} 4-8 & -4 \\ 0 & 2+2\alpha-4\alpha \end{pmatrix} = \begin{pmatrix} -4 & -4 \\ 0 & 2-2\alpha \end{pmatrix}$$

$\rightsquigarrow \lambda_1 = -4 < 0$
$\lambda_2 = 2(1-\alpha)$

$P_4(4,0)$ is a { saddle if $0<\alpha<1$; stable node $\alpha>1$ }

2.3) $\alpha = \alpha_0 - \frac{1}{4} = 1 - \frac{1}{4} = 3/4$

$$\begin{cases} \dot{x} = x(4-x-y) \\ \dot{y} = y\left(3.5 - y - \frac{3x}{4}\right) \end{cases}$$



For long time, the two species coexist for $\alpha < \alpha_0$ at $(2,2)$

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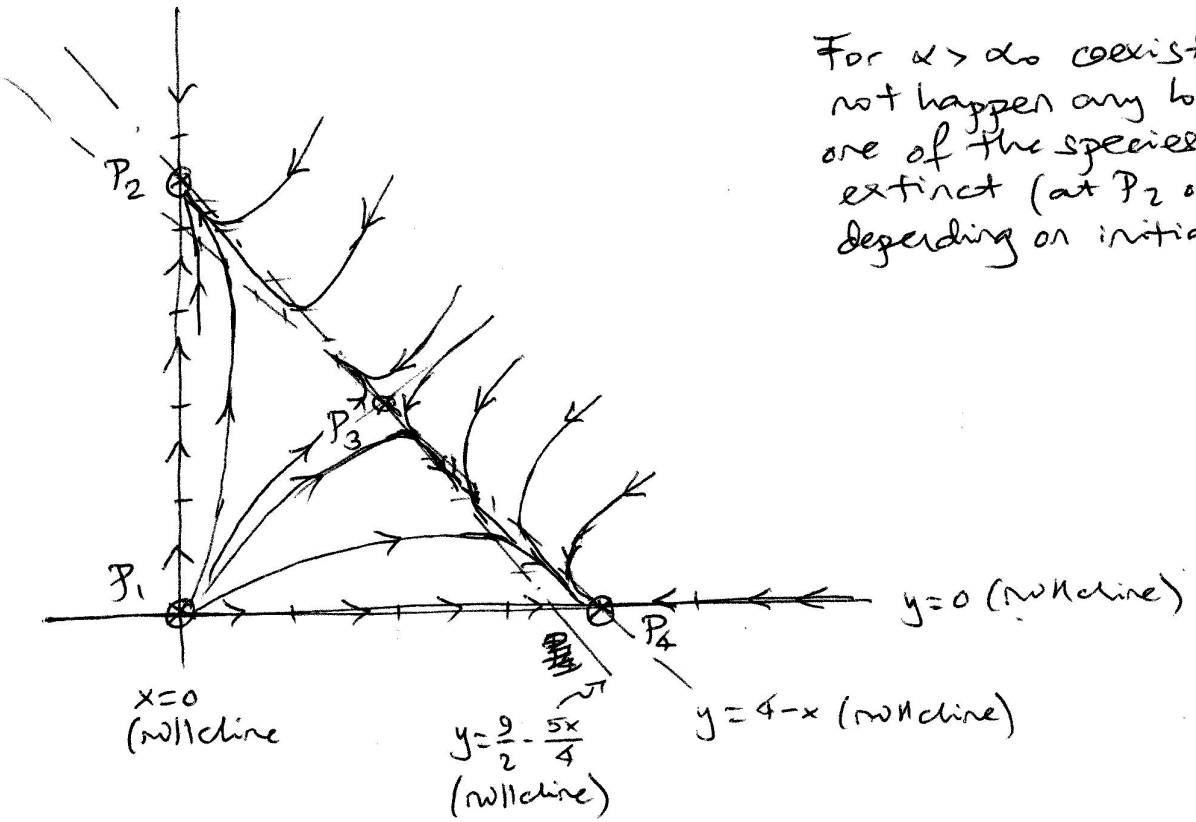
$\alpha = \alpha_0 + \frac{1}{4} = \frac{5}{4}$

$\dot{x} = x(4 - x - y)$

$\dot{y} = y\left(\frac{9}{2} - y - \frac{5x}{4}\right)$

at (2,2)

For $\alpha > \alpha_0$ coexistence does not happen any longer and one of the species becomes extinct (at P_2 or P_4), depending on initial conditions



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PROBLEM 3. (2.5 points)

a) Split the following problem into two one variable problems:

$$\begin{cases} u_t - 2u_{xx} = 2u, & -\frac{\pi}{2} < x < \frac{\pi}{2}, \quad t > 0, \\ u(-\pi/2, t) = u(\pi/2, t), & t > 0, \\ u_x(-\pi/2, t) = u_x(\pi/2, t), & t > 0, \\ u(x, 0) = 8 - 3\sin(2x) - 8\cos^2(2x), & -\frac{\pi}{2} < x < \frac{\pi}{2}. \end{cases}$$

b) Solve the problems you have obtained.

c) Find the solution of the full problem.

Solution:

a) Separating variable $u(x,t) = \phi(x)G(t)$ we find that:

$$\begin{cases} \phi''(x) + \lambda\phi(x) = 0, \\ \phi(-\pi/2) = \phi(\pi/2), \\ \phi'(-\pi/2) = \phi'(\pi/2), \end{cases} \qquad G'(t) = 2(1-\lambda)G(t).$$

b) Thanks to the periodic boundary conditions we know that:

$$\lambda_n = \left(\frac{n\pi}{\pi/2}\right)^2 = 4n^2, \quad n = 0, 1, 2, \ldots \qquad \phi_n(x) = c_1\cos(2nx) + c_2\sin(2nx)$$

Then,

$$G'(t) = 2(1-4n^2)G(t) \quad \Longrightarrow \quad G(t) = Ke^{2(1-4n^2)t}.$$

c) The general solution comes after applying the superposition principle:

$$u(x,t) = A_0 e^{2t} + \sum_{n=1}^{\infty} e^{(2-8n^2)t}\left(A_n\cos(2nx) + B_n\sin(2nx)\right)$$

Moreover, using the initial data we are able to compute the coefficients of the previous series:

$$\begin{aligned} u(x,0) = A_0 + \sum_{n=1}^{\infty}\left(A_n\cos(2nx) + B_n\sin(2nx)\right) &= 8 - 3\sin(2x) - 8\cos^2(2x) \\ &= 8 - 3\sin(2x) - 4(1+\cos(4x)) \end{aligned}$$

such that $B_n = 0, \quad n \neq 1, \quad A_n = 0, \quad n \neq 0, n \neq 2$ y:

$$A_0 = 4, \quad B_1 = -3, \quad A_2 = -4.$$

Therefore, the solution is:

$$u(x,t) = 4e^{2t} - 3e^{-6t}\sin(2x) - 4e^{-30t}\cos(4x).$$

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PROBLEM 4

4.1) $$u(r,\theta) = \frac{U(r)}{r} P(\theta)$$

$$0 = \nabla^2 u = \frac{1}{r}\frac{\partial^2}{\partial r^2}(ru) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial u}{\partial\theta}\right)$$

$$= \frac{1}{r} P U'' + \frac{1}{r^2 \sin\theta}\frac{U}{r}\left(\sin\theta\, P'\right)' = 0 \rightarrow \text{multiplying by } r^3 \frac{1}{PU},$$

$$\underbrace{\frac{1}{P\sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{dP}{d\theta}\right)}_{\text{dep. } \theta \text{ only}} = \underbrace{-\frac{r^2 U''}{U}}_{\text{dep. } r \text{ only}} = -\lambda$$

Two differential equations:

$$\frac{dU''}{dr^2} - \frac{\lambda}{r^2} U = 0 \qquad (4.1.1)$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{dP}{d\theta}\right) + \lambda P = 0 \qquad (4.1.2)$$

4.2) Eq. (4.1.2) is the one given in the exercise, hence for $P(\theta)$ to be differentiable we have $\lambda = n(n+1) = \lambda_n$ $\rightsquigarrow$ solution is $P_n(\theta)$. $n \geq 0$

With this, we solve (4.1.1) assuming $U(r) = r^\alpha$; thus,

$$\alpha(\alpha-1) r^{\alpha-2} - n(n+1) r^{\alpha-2} = 0 \Rightarrow \alpha^2 - \alpha - n(n+1) = 0$$

$$\alpha = \frac{1 \pm \sqrt{1 + 4n^2 + 4n}}{2} = \frac{1 \pm (2n+1)}{2} = \begin{cases} n+1 \\ -n \end{cases}$$

and we have two L.I. solutions, i.e.

$$U_n(r) = r^{n+1} \quad \text{and} \quad U_n(r) = \frac{1}{r^n}$$

4.3) Eq. (4.1.2) is a (singular) Sturm-Liouville equation of the standard form (independent variable θ)

$$\frac{d}{d\theta}\left(p(\theta)\frac{d\phi}{d\theta}\right) + q(\theta)\phi(\theta) + \lambda\sigma(\theta)\phi(\theta) = 0$$

where $p(\theta) = \sin\theta$, $q(\theta) = 0$, and $\sigma(\theta) = \sin\theta$.

To find the solution of $\nabla^2 u = 0$, we use a linear superposition principle and write

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$$v(r,\theta) = \frac{1}{r}\sum_{n=0}^{\infty}\left(A_n r^{n+1} + B_n \frac{1}{r^n}\right) P_n(\theta) =$$

$$= \sum_{n=0}^{\infty}\left(A_n r^n + B_n \frac{1}{r^{n+1}}\right) P_n(\theta)$$

For the solution to be well defined at the origin, we request

$$B_n = 0 \quad \text{(regularity condition)}$$

Finally, imposing $v(R,\theta) = V(\theta)$ we have

$$\sum_{n=0}^{\infty} A_n R^n P_n(\theta) = \sum_{n=0}^{\infty} V_n P_n(\theta),$$

where

$$\boxed{V_n = \frac{\langle P_n, V\rangle}{\langle P_n, P_n\rangle} = \frac{\int_0^{\pi} d\theta\, P_n(\theta)\, V(\theta) \sin\theta}{\int_0^{\pi} d\theta\, P_n^2(\theta) \sin\theta}}$$

and

$$A_n = \frac{V_n}{R^n},$$

since by the Sturm-Liouville theorem the $\{P_n(\theta)\}$ are an orthogonal basis of eigenfunctions using the scalar product

$$\langle f, g\rangle = \int_0^{\pi} d\theta \underbrace{\sin\theta}_{\sigma(\theta)} f(\theta) g(\theta).$$

Finally,

$$\boxed{v(r,\theta) = \sum_{n=0}^{\infty} V_n \left(\frac{r}{R}\right)^n P_n(\theta)}$$

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