ADVANCED MATHEMATICS

Bachelor Degree in Aerospace Engineering, Universidad Carlos III de Madrid

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Total duration: 4 hours

PROBLEM 1. (2.5 points) Consider the following first order linear equation:

$$xy'' + (1-2x)y' + (x-1)y = (x+1)e^x\,, \qquad x > 0\,.$$

(i) Using that $y_1(x) = e^x$ is a solution to the corresponding homogeneous equation, find a second, linearly independent, solution [of the type $y_2(x) = v(x)y_1(x)$] and thus obtain the general solution of the corresponding homogeneous equation.

(ii) Find the general solution of the full, non-homogeneous equation.

PROBLEM 2. (2.5 points) Consider two biological species described by the system [with $\alpha > 0$ and $x(t), y(t) \geq 0$]

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = x(4 - x - y), \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = y(2 + 2\alpha - y - \alpha x). \end{cases}$$

2.1 Discuss qualitatively the mechanisms which determine the time evolution of each species.

2.2 Find the critical points and classify them for $0 < \alpha < \alpha_0$ and $\alpha_0 < \alpha$, where α_0 (to be found) is the value at which there is a change in the type of the critical point that lies in the interior $(x, y > 0)$ region of phase space.

2.3 Sketch the phase portraits for $\alpha = \alpha_0 - 1/4$ and $\alpha = \alpha_0 + 1/4$. How does a change in α affect the long-time evolution of the system?

PROBLEM 3. (2.5 points) Solve the following problem for the heat equation with an internal heat source:

$$\begin{cases} u_t = 2u_{xx} + 2u, & -\dfrac{\pi}{2} < x < \dfrac{\pi}{2}, \quad t > 0, \\ u(-\pi/2, t) = u(\pi/2, t), & t > 0, \\ u_x(-\pi/2, t) = u_x(\pi/2, t), & t > 0, \\ u(x, 0) = 8 - 3\sin(2x) - 8\cos^2(2x), & -\dfrac{\pi}{2} < x < \dfrac{\pi}{2}. \end{cases}$$

PROBLEM 4. (2.5 points)

Consider the 3D Laplace equation on a solid sphere, for a function $u(r, \theta)$ which only depends on *two spherical coordinates*, the distance to the origin (r) and the angle with the z-axis (θ):

$$\nabla^2 u = \frac{1}{r}\frac{\partial^2}{\partial r^2}(ru) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) = 0, \quad 0 < r < R, \quad 0 < \theta < \pi.$$

4.1 By separating variables as $u(r, \theta) = \dfrac{U(r)}{r}P(\theta)$, obtain differential equations for $U(r)$ and $P(\theta)$.

4.2 Solve the equations in 4.1 knowing that $\dfrac{1}{\sin\theta}\dfrac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\dfrac{\mathrm{d}P}{\mathrm{d}\theta}\right) + \lambda P = 0$ only admits differentiable solutions provided that $\lambda = n(n+1)$ with $n \geq 0$; such solutions $[P_n(\theta)]$ are called Legendre polynomials.

4.3 By identifying all elements of the relevant Sturm-Liouville problem, find the solution of the original PDE, subject to the boundary condition $u(R, \theta) = V(\theta)$, where $V(\theta)$ is a known function.
