Practice Material for Final Differential Equations

Note that this list of questions is not exhaustive. You should also be prepared to answer questions such as those on test 1, test 2, the make up test for test 2, inclass quizzes, and homework assignments. A copy of p. 557 from the text will be provided with the final exam.

Write your name on the back of your test. Show all work. Where possible test your solutions.

1. Solve the initial value problem of the differential equation:

$$\frac{dy}{dt} = \frac{2t}{1+t^2}y + \frac{2}{1+t^2}, \qquad y(0) = -2$$

Solution: This is a first order linear differential equation, so we employ the method of integrating factors. First we rearrange the equation to put the unknown function and its derivatives on the left-hand-side,

$$\frac{dy}{dt} - \frac{2t}{1+t^2}y = \frac{2}{1+t^2}.$$

Multiply both sides by the integrating factor

$$e^{\int(-(2t)/(1+t^2))dt} = e^{-\ln(1+t^2)} = \frac{1}{1+t^2},$$

$$\frac{1}{1+t^2}(\frac{dy}{dt} - \frac{2t}{1+t^2}y) = \frac{2}{(1+t^2)^2}.$$

Integrating both sides gives

$$\frac{y}{1+t^2} = \int \frac{2}{(1+t^2)^2}dt.$$

We use a trigonometric substitution to evaluate the right-hand-side. Let $t = \tan\theta$, then $dt = \sec^2\theta d\theta$.

$$\begin{aligned}
\int \frac{2}{(1+t^2)^2}dt &= \int 2\cos^4\theta\sec^2\theta d\theta \\
&= \int 2\cos^2\theta d\theta \\
&= \int (1+\cos 2\theta)d\theta \\
&= \theta + \frac{\sin 2\theta}{2} + C \\
&= \theta + \sin\theta\cos\theta + C \\
&= \tan^{-1}t + \sin\tan^{-1}t\cos\tan^{-1}t + C \\
&= \tan^{-1}t + \frac{t}{(1+t^2)} + C.
\end{aligned}$$

So

$$\frac{y}{1+t^2} = \int \frac{2}{(1+t^2)^2}dt$$

$$
\begin{aligned}
&= \tan^{-1} t + \frac{t}{(1+t^2)} + C \\
y(t) &= (1+t^2)\tan^{-1} t + t + C(1+t^2).
\end{aligned}
$$

From the initial condition,

$$
\begin{aligned}
y(0) = -2 &= \tan^{-1} 0 + 0 + C(1+0^2) \\
C &= -2.
\end{aligned}
$$

The solution is

$$
y(t) = (1+t^2)\tan^{-1} t + t - 2(1+t^2).
$$

Check:

$$
\begin{aligned}
y'(t) &= 1 + 2t\tan^{-1} t + 1 - 4t \\
&= \frac{2t}{1+t^2}((1+t^2)\tan^{-1} t + t - 2(1+t^2)) + \frac{2}{1+t^2}.
\end{aligned}
$$

2. A 5-gallon bucket is full of pure water. Suppose we begin dumping salt into the bucket at a rate of 1/4 pounds per minute. Also, we open the spigot so that 1/2 gallons per minute leaves the bucket, and we add pure water to keep the bucket full. If the saltwater solution is always well mixed, what is the amount of salt in the bucket after

(a) one minute? (b) 10 minutes? (c) 60 minutes? (d) 1000 minutes? (e) a very long time?

Solution: The rate of change in the amount of salt in the bucket is dS/dt, the amount that comes in per unit time minus what goes out per unit time, so

$$
\frac{dS}{dt} = \frac{1}{4} - \frac{S}{10}
$$

since 1/2 gallon is one tenth of what the bucket holds, one tenth of the salt in the bucket leaves the bucket each minute. This can be solved via separation of variables.

$$
\begin{aligned}
\frac{1}{1/4 - S/10} dS &= dt \\
\int \frac{10}{5/2 - S} dS &= \int dt \\
-10\ln(5/2 - S) &= t + C \\
\ln(5/2 - S) &= -t/10 - C/10 \\
e^{\ln(5/2-S)} &= e^{-t/10 - C/10} \\
(5/2 - S) &= Ke^{-t/10}.
\end{aligned}
$$

From the initial condition, $K = e^{-C/10} = 5/2$, so $S(t)$ is

$$S(t) \;=\; 5/2(1 - e^{-t/10}).$$

a) $S(1) = 5/2(1 - e^{-1/10}) \approx .2379$ lbs. b) $S(10) = 5/2(1 - e^{-1}) \approx 1.5803$ lbs. c) $S(60) = 5/2(1 - e^{-6}) \approx 2.4938$ lbs. d) $S(1000) = 5/2(1 - e^{-100}) \approx 2.5$ lbs. e) $\lim_{t\to\infty} S(t) = 5/2$ lbs.

3. Suppose $dy/dt = f(y)$ has an equilibrium point at $y = y_0$ and

(a) $f'(y_0) = 0$, $f''(y_0) = 0$, and $f'''(y_0) > 0$; Is y_0 a source, a sink, or a node?

Solution: Since the first two derivatives are zero, and the third derivative is positive, Taylor's theorem implies that $f(y)$ is approximately

$$\frac{f'''(y_0)}{3!}(y - y_0)^3$$

for y near y_0. $f(y)$ is increasing near y_0 (negative to the left of y_0 and positive to the right), so y_0 is a source.

(b) $f'(y_0) = 0$, $f''(y_0) = 0$, and $f'''(y_0) < 0$; Is y_0 a source, a sink, or a node?

Solution: The function $f(y)$ will be approximately that given in part a) for y close to y_0, so $f(y)$ is decreasing near y_0 (positive to the left of y_0 and negative to the right), therefore y_0 is a sink.

(c) $f'(y_0) = 0$ and $f''(y_0) > 0$; Is y_0 a source, a sink, or a node?

Solution: Taylor's theorem implies that $f(y)$ is approximately

$$\frac{f''(y_0)}{2!}(y - y_0)^2$$

for y near y_0. $f(y)$ is positive on both sides of y_0, so y_0 is a node.

4. Find the solution of the differential equation satisfying the given initial condition using Laplace transforms:

$$\frac{d^2y}{dt^2} + 9y = u_2(t)\cos(4(t-2)), \qquad y(0) = 0, \qquad y'(0) = 3.$$

Solution: Taking Laplace transforms

$$s^2\mathcal{L}[y] - sy(0) - y'(0) + 9\mathcal{L}[y] = e^{-2s}\frac{s}{s^2+16}.$$

Solving for the Laplace transform of the solution, and computing the partial fractions decomposition,

$$\begin{aligned}\mathcal{L}[y] &= \frac{1}{s^2+9}(3 + e^{-2s}\frac{s}{s^2+16}) \\ &= \frac{3}{s^2+9} - \frac{se^{-2s}}{7(s^2+16)} + \frac{se^{-2s}}{7(s^2+9)}.\end{aligned}$$

The inverse Laplace transform is

$$y(t) \quad = \quad \sin 3t - u_2(t)\cos(4(t-2))/7 + u_2(t)\cos(3(t-2))/7.$$

5. For the system given below,

(a) Find and classify all equilibria in the first quadrant

(b) Sketch the phase plane of the system near each equilibrium point

(c) describe the behavior of solutions near each equilibrium point

$$\begin{aligned} \frac{dx}{dt} &= x(2-x-y) \\ \frac{dy}{dt} &= y(y-x) \end{aligned}$$

Solution:

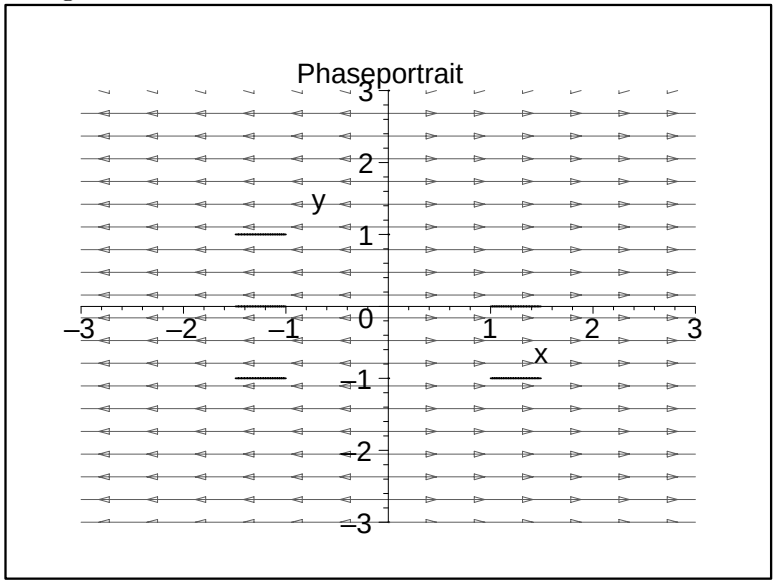
(a) The equilibrium solutions are $(0,0)$, $(2,0)$ and $(1,1)$. To classify these points, we compute the Jacobian matrix at each point.

$$\begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} \quad = \quad \begin{pmatrix} 2-2x-y & -x \\ -y & 2y-x \end{pmatrix}$$

The Jacobians are: for $(0,0)$

$$\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$$

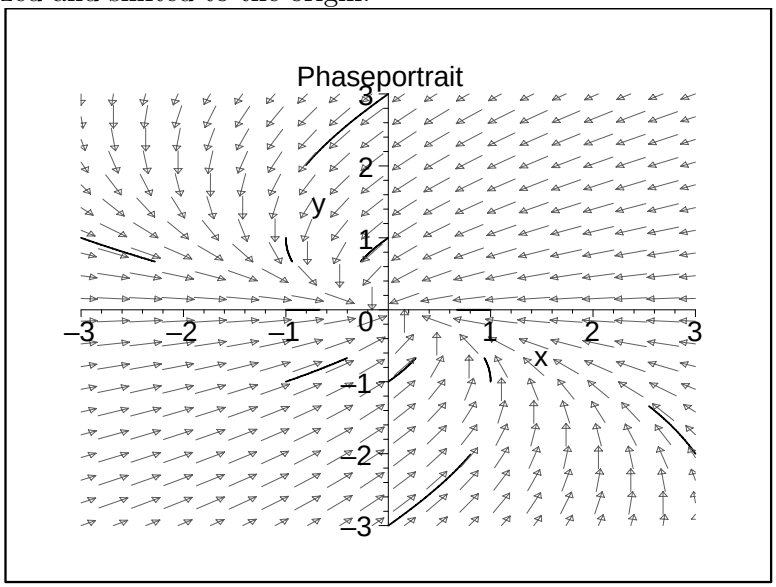
with eigenvalues $\lambda = 2, 0$; The long term behavior of an equilibrium point of a nonlinear system with a zero eigenvalue can differ from its linearization.



for $(2, 0)$

$$\begin{pmatrix} -2 & -2 \\ 0 & -2 \end{pmatrix}$$

with repeated eigenvalues $\lambda = -2$, sink; the phase portrait below shows this equilibrium point linearized and shifted to the origin.



for $(1, 1)$

$$\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}$$

with eigenvalues $\lambda = \pm\sqrt{2}$, saddle; the phase portrait below shows this equilibrium point linearized and shifted to the origin.

