Problem 1

Consider a spherical container of radius R which is punctured at its south pole, so that fluid escapes due to gravity. Hence, the volume, $V(t)$, of the remaining fluid at time t decreases as

$$\frac{dV}{dt} = -av,$$

where v is the fluid speed and $a > 0$ is a known constant. Torricelli's law states that $v = \sqrt{2gy}$, where y is the highest level of the remaining fluid, measured from the south pole of the container, and g is the acceleration of gravity. Finally, note that, for this solid of revolution,

$$V = \int_0^y A(s)\, ds, \quad \text{where} \quad A(y) = \pi r^2 = \pi(R^2 - (y-R)^2).$$

1.1 Given that the container is completely full at $t = 0$, find an implicit form of the function $y(t)$ describing the highest level of the remaining fluid.
(HINT: Note that $\frac{dV}{dt} = \frac{dV}{dy}\frac{dy}{dt}$.)

1.2 Compute the time required to empty the container completely.

1.1 Thanks to the previous expressions and Torricelli's law, we find that

$$\frac{dV}{dt} = -a\sqrt{2g}y^{1/2}.$$

Also, using the chain rule,

$$\frac{dV}{dt} = \frac{dV}{dy}\frac{dy}{dt}.$$

Substituting these two equalities into the equation (after using the integral expression for the volume) we arrive at

$$A(y)\frac{dy}{dt} = -a\sqrt{2g}y^{1/2},$$

hence,

$$\pi(2Ry^{1/2} - y^{3/2})dy = -a\sqrt{2g}dt.$$

Integrating

$$2\pi R\frac{y^{3/2}}{3/2} - \pi\frac{y^{5/2}}{5/2} = -a\sqrt{2g}t + k.$$

The initial condition at $t = 0$ implies $y(0) = 2R$, so that

$$k = \pi(2R)^{5/2}\left(\frac{4}{3} - \frac{2}{5}\right) \Rightarrow k = \frac{14}{15}\pi(2R)^{5/2},$$

and the (implicit) solution to the ODE for $y(t)$ is

$$2\pi R\frac{y^{3/2}}{3/2} - \pi\frac{y^{5/2}}{5/2} = -a\sqrt{2g}t + \frac{14}{15}\pi(2R)^{5/2}.$$

1.2 Assuming $y = 0$ we obtain the time necessary to empty the container, i.e.,

$$t = \frac{14}{15}\frac{\pi(2R)^{5/2}}{a\sqrt{2g}}.$$

Problem 2

$$\ddot{x} - 2\dot{x} + x = \frac{e^t}{1+t^2} \tag{1}$$

Homogeneous equation:

$$\ddot{x} - 2\dot{x} + x = 0; x = e^{rt} \to r^2 - 2r + 1 = 0 \to r = 1, \tag{2}$$

hence we have a double root. The solution set is spanned by

$$\left\{e^t, te^t\right\}. \tag{3}$$

Therefore, the general solution of the homogeneous equation reads

$$x_H(t) = c_1 + c_2 te^t. \tag{4}$$

To solve the complete problem, we use the method of variation of parameters. Substituting a solution of the form $x(t) = u_1(t)e^t + u_2(t)te^t$, after differentiation we arrive at two equations to determine u_1, u_2:

$$\begin{aligned} \dot{u}_1 e^t + \dot{u}_2 te^t &= 0, \\ \dot{u}_1 e^t + \dot{u}_2 (1+t)e^t &= \frac{e^t}{1+t^2}. \end{aligned} \tag{5}$$

We determine the Wronskian:

$$\begin{vmatrix} e^t & te^t \\ e^t & (1+t)e^t \end{vmatrix} = e^{2t}. \tag{6}$$

For each function $u_{1,2}(t)$, we thus get

$$\dot{u}_1 = \frac{\begin{vmatrix} 0 & te^t \\ \frac{e^t}{1+t^2} & (1+t)e^t \end{vmatrix}}{e^{2t}} = -\frac{t}{1+t^2} \to u_1(t) = -\frac{1}{2}\ln(1+t^2). \tag{7}$$

$$\dot{u}_2 = \frac{\begin{vmatrix} e^t & e^t \\ 0 & \frac{e^t}{1+t^2} \end{vmatrix}}{e^{2t}} = \frac{1}{1+t^2} \to u_2(t) = \arctan t. \tag{8}$$

Finally,

$$x(t) = c_1 + c_2 te^t - \frac{1}{2}e^t \ln(1+t^2) + te^t \arctan t. \tag{9}$$

PROBLEM 3

3.1) Homogeneous system:

$$\left.\begin{aligned}\dot{x} &= x - y\\ \dot{y} &= 4x - 3y\end{aligned}\right\} \quad \frac{d\vec{X}}{dt} = \underbrace{\begin{pmatrix}1 & -1\\ 4 & -3\end{pmatrix}}_{A}\vec{X} \qquad \vec{X} = \begin{pmatrix}x\\ y\end{pmatrix}$$

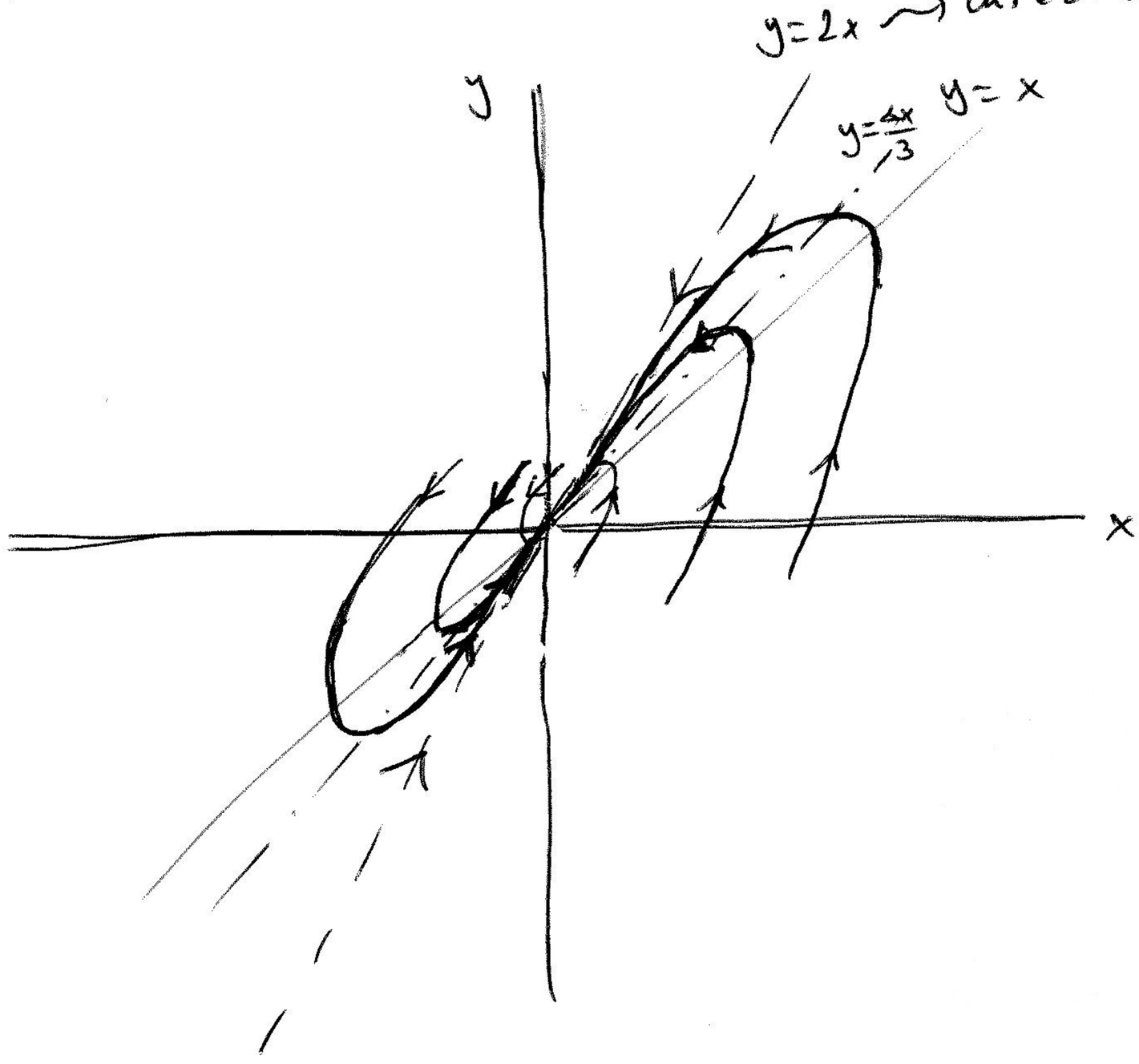
$$\det(A - \lambda\mathbb{I}) = (1-\lambda)(-3-\lambda) + 4 = \lambda^2 - \lambda + 3\lambda - 3 + 4$$
$$= \lambda^2 + 2\lambda + 1 = (\lambda+1)^2 = 0 \Rightarrow \lambda = -1 \text{ double}$$

A is not a multiple of the identity matrix

$\Rightarrow$ A is defective and the origin is a stable improper node.

We look for the eigenvector of eigenvalue $\lambda = -1$:

$$\underbrace{\begin{pmatrix}2 & -1\\ 4 & -2\end{pmatrix}}_{A-(-1)\mathbb{I}}\begin{pmatrix}a\\ b\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix} \rightsquigarrow \left.\begin{aligned}b &= 2a\\ 2b &= 4a\end{aligned}\right\} \rightsquigarrow \vec{v} = \begin{pmatrix}1\\ 2\end{pmatrix}$$



$x = y \Rightarrow \dot{x} = 0$ (vertical flow)

$y = \frac{4}{3}x \Rightarrow \dot{y} = 0$ (horizontal flow

3.2) $\dot{x} = x - y + t \Rightarrow y = x - \dot{x} + t \rightsquigarrow \dot{y} = \dot{x} - \ddot{x} + 1$

$\dot{y} = 4x - 3y - 1 \qquad \dot{y} = 4x - 3y - 1$

$\Rightarrow \quad \dot{x} - \ddot{x} + 1 = 4x - 3(x - \dot{x} + t) - 1$

$\dot{x} - \ddot{x} + 1 = 4x - 3x + 3\dot{x} - 3t - 1$

$\ddot{x} + 2\dot{x} + x = 2 + 3t$

Homogeneous eq.:

$\ddot{x} + 2\dot{x} + x = 0 \qquad k^2 + 2k + 1 = 0 \rightsquigarrow (k+1)^2 = 0$

$x = e^{kt} \qquad k = -1$ double

$\Rightarrow x_h(t) = A e^{-t} + B t e^{-t}$

Particular solution:

$x_p(t) = a + bt$

$\dot{x}_p = b$

$\ddot{x}_p = 0$

$\ddot{x}_p + 2\dot{x}_p + x_p = 2 + 3t$

$0 + 2b + a + bt = 2 + 3t$

$\rightsquigarrow \quad 2b + a = 2 \quad a = 2 - 6 = -4$

$b = 3$

$$x(t) = A e^{-t} + B t e^{-t} - 4 + 3t$$

$\dot{x} = -A e^{-t} + B e^{-t} - B t e^{-t} + 3$

$$y(t) = x - \dot{x} + t = A e^{-t} + B t e^{-t} - 4 + 3t + A e^{-t} - B e^{-t} + B t e^{-t} - 3 + t$$

$$= 2A e^{-t} + 2B t e^{-t} - B e^{-t} - 7 + 4t$$

Problem 4

4.1 Compute the eigenvalues and eigenfunctions of the problem $\begin{cases} u'' - u' + \lambda u = 0, & 0 < x < \pi, \\ u(0) = u(\pi) = 0. \end{cases}$

4.2 Write the equation in Sturm-Liouville form using an integrating factor.

4.3 Obtain the orthogonality relations for the eigenfunctions.

4.1 We consider solutions of the form: e^{rx} and obtain the characteristic polynomial:

$$r^2 - r + \lambda u = 0 \quad \Longrightarrow \quad r = \frac{1 \pm \sqrt{1 - 4\lambda}}{2} = \frac{1}{2} \pm \sqrt{\frac{1}{4} - \lambda}.$$

Case 1: $\lambda = \dfrac{1}{4}$ is not an eigenvalue because:

$$u = \mathrm{e}^{x/2}(C_1 + C_2 x), \quad u(0) = C_1 = 0, \quad u(\pi) = \mathrm{e}^{\pi/2} C_2 \pi = 0 \implies C_2 = 0.$$

Case 2: $\lambda < \dfrac{1}{4}$:

$$u = \mathrm{e}^{x/2}\left(C_1 \cosh\left(x\sqrt{\frac{1}{4} - \lambda}\right) + C_2 \sinh\left(x\sqrt{\frac{1}{4} - \lambda}\right)\right)$$

$$u(0) = C_1 = 0, \quad u(\pi) = \mathrm{e}^{\pi/2} C_2 \sinh\left(\pi\sqrt{\frac{1}{4} - \lambda}\right) = 0 \quad \Longrightarrow \quad C_2 = 0.$$

Hence, we do not obtain eigenvalues in this case either.

Case 3: $\lambda > \dfrac{1}{4}$:

$$u = \mathrm{e}^{x/2}\left(C_1 \cos\left(x\sqrt{\lambda - \frac{1}{4}}\right) + C_2 \sin\left(x\sqrt{\lambda - \frac{1}{4}}\right)\right)$$

First we obtain $u(0) = C_1 = 0$ and then

$$u(\pi) = \mathrm{e}^{\pi/2} C_2 \sin\left(\pi\sqrt{\lambda - \frac{1}{4}}\right) = 0 \quad \Longrightarrow \quad \begin{cases} \lambda_n = n^2 + \dfrac{1}{4}, & n = 1, 2, \ldots \\ \\ u_n(x) = \mathrm{e}^{x/2}\sin(nx), & n = 1, 2, \ldots \end{cases}$$

4.2 A Sturm-Liouville equation has the form

$$\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + q(x)u + \lambda\sigma(x)u = 0, \implies p(x)u'' + p'(x)u' + q(x)u + \lambda\sigma(x)u = 0.$$

We multiply our equation by an integrating factor to obtain a Sturm-Liouville equation:

$$\mu(x)u'' - \mu(x)u' + \mu(x)\lambda u = 0 \implies \mu(x) = p(x) = -p'(x) \implies \frac{p'(x)}{p(x)} = -1 \implies \mu(x) = \mathrm{e}^{-x},$$

obtaining

$$\mathrm{e}^{-x}u'' - \mathrm{e}^{-x}u' + \mathrm{e}^{-x}\lambda u = 0,$$

then $p(x) = \mathrm{e}^{-x}$, $\sigma(x) = \mathrm{e}^{-x}$, $q(x) = 0$.

4.3 The eigenfunctions are orthogonal with respect to the weight $\sigma(x)$ on the interval $[0, \pi]$, so that the orthogonality relation for $n \neq m$ is:

$$\int_0^\pi \mathrm{e}^{-x} u_n(x) u_m(x) dx = \int_0^\pi \mathrm{e}^{-x}\mathrm{e}^{x/2}\sin(nx)\mathrm{e}^{x/2}\sin(mx)dx = \int_0^\pi \sin(nx)\sin(mx)dx = 0.$$

Problem 5

The homogeneous problem is

$$u_t = u_{xx}. \tag{1}$$

By separation of variables,

$$u(x,t) = X(x)T(t), \tag{2}$$

we arrive at the following boundary-value problem (BVP):

$$X'' + \lambda X = 0; X(0) = X(\pi) = 0, \tag{3}$$

where the eigenvalues are $\lambda_n = n^2$ and the eigenfunctions are $X_n(x) = \sin(nx)$.

The solution reads $u(x,t) = \sum_{n=1}^{\infty} T_n(t)X_n(x)$. In addition, the forcing has to be expressed in terms of the eigenfunctions solving the BVP, $1 + xt = \sum_{n=1}^{\infty} \gamma_n(t)X_n(x)$. Substituting into the original problem,

$$\sum_{n=1}^{\infty} T_n' X_n = \sum_{n=1}^{\infty} T_n(-\lambda X_n) + \sum_{n=1}^{\infty} \gamma_n X_n \rightarrow T_n' = -\lambda T_n + \gamma_n. \tag{4}$$

The solution is of the form

$$T_n(t) = T_n(0)e^{-\lambda_n t} + \int_0^t e^{-\lambda_n(t-s)}\gamma_n(s)ds, \tag{5}$$

where $\gamma_n, T_n(0)$ are to be determined:

$$\gamma_n = \int_0^{\pi} (1+xt)\sin(nx)dx = -\frac{\pi}{n}t(-1)^n, \tag{6}$$

$$u(x,0) = \sum_{n=1}^{\infty} T_n(0)\sin(nx) \rightarrow T_1(0) = 1; T_n(0) = 0 \ (n \neq 1), \tag{7}$$

$$T_n(t) = T_n(0)e^{-n^2 t} + \int_0^t e^{-n^2(t-s)} \left(-\frac{\pi}{n}s(-1)^n\right) ds. \tag{8}$$

Upon substitution, we arrive at the final solution:

$$u(x,t) = e^{-t}\sin x + \sum_{n=1}^{\infty} \left[-\frac{\pi}{n^3}(-1)^n t\sin(nx) + \frac{\pi}{n^5}\left(1 - e^{-n^2 t}\right)\sin(nx)\right]. \tag{9}$$