UC3M Fakefinal, December 21, 2016

Advanced Mathematical Methods

Name: _____.

Exercise 1

A 5-gallon bucket is full of pure water. Suppose we begin by dumping salt into the bucket at a rate of $\frac{1}{4}$ pounds per minute. Also we open the spigot so that $\frac{1}{2}$ gallons per minute leaves the bucket, and we add pure water to keep the bucket full. If the salt water solution is always well mixed establish a differential equation for the amount of salt in the bucket. What are the independent and the dependent variable and the parameters? What is the amount of salt in the bucket after

- 1 minute?
- 1000 minutes?
- a very long time?

Exercise 2

Obtain the general solution to

$$x' = 2x - 5y + \frac{1}{\sin t},$$
$$y' = x - 2y + \frac{1}{\cos t},$$

for $\frac{\pi}{2} < t < \pi$.

Exercise 3

Determine the equilibrium points and classify them according to the linearised system. Will this model correctly the non-linear system?

$$\dot{x} = 1 - y, \dot{y} = x^2 - y^2.$$

Exercise 4

- Write down the most general PDE of third order with two independent variables.
- State whether the following equation is nonlinear, linear homogeneous or linear inhomogeneous: $-u_{xx} - 4u_{yy} - u_{tt} - 2u_{zz} = -m^2 u$, where m is a constant. What *type* of second order equations is it? Provide reasons in both cases.

Exercise 5

Find the formal solution to the problem

$$y'' + \omega^2 y = f(t),\ y(0) = 1, y'(0) = 0,$$

where f is periodic with period 2 and

$$f = \left\{ \begin{array}{ll} 1 - t, & \text{for } 0 \leq t < 1 \\ -1 + t & \text{for } 1 \leq t < 2 \end{array} \right\}$$

Exercise 6

Solve

$$u_t = u_{xx} + 1 - |1 - 2x|$$

with $u(0, t) = u(1, t) = u(x, 0) = 0$.

Exercise 7

1. Give the definition of a Lipschitz continuous function and give an example of a function that satisfies this condition by showing explicitly that the function satisfies the condition.

2. What are the assumptions of the Fourier theorem and what is the main statement?

Exercise 8

1. Suppose you have a machine whose vibrations can be accurately modeled with a harmonic oscillator with a harmonic oscillator with almost no damping but you don't know the mass or spring constant. If you can observe the vibrations of the machine under external forcing given by $\cos \omega t$ for various values of ω, what can you say about the mass and the spring constant?

2. Show that the derivative of an odd function is even and that the derivative of an even function is odd.