AEROSPACE ENGINEERING.
Advanced Mathematics. Final exam. January 21st, 2016.

Select and solve four out of the following five problems.

1. Given Frommer's system of ODEs (invariant under $x \to -x$, $t \to -t$, $y \to y$):

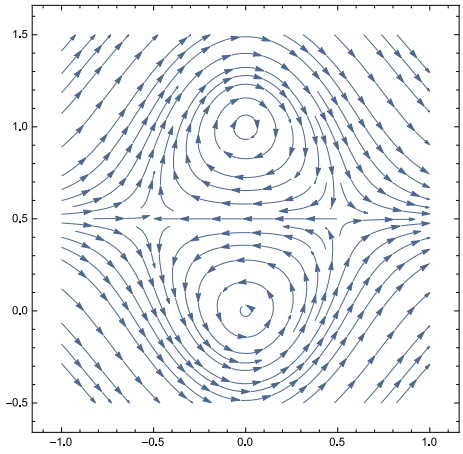
$$\frac{dy}{dt} = x - 2xy, \quad \frac{dx}{dt} = -y + x^2 + y^2,$$

find, classify and ascertain the stability of its equilibrium points (constant solutions) in the phase plane (x, y).

Solution. The E.P. solve the equations $x(1-2y) = 0$, $-y + x^2 + y^2 = 0$. *We get* $(0,0)$, $(0,1)$, $(\frac{1}{2}, \frac{1}{2})$, $(-\frac{1}{2}, \frac{1}{2})$. *The Jacobian matrix about the E.P.* (x_0, y_0) *is*

$J(x_0, y_0) = \begin{pmatrix} 2x_0 & 2y_0 - 1 \\ 1 - 2y_0 & -2x_0 \end{pmatrix}$. *The sum of off-diagonal terms is identically* 0.

Then the E.P. $(\frac{1}{2}, \frac{1}{2})$ *and* $(-\frac{1}{2}, \frac{1}{2})$ *are* <u>unstable</u> *saddle points and* $(0,0)$ *and* $(0,1)$ *are centers of the linearized system of ODEs. They are also centers of the nonlinear system of ODEs and therefore* <u>stable</u>. *Why? The line* $y = \frac{1}{2}$, $\dot{x} = x^2 - \frac{1}{4}$ *comprises*



two E.P. and separatrices that enter and exit them. The other separatrices leave and enter the saddle points parallel to the y *axis. Using the symmetry, a separatrix leaves* $(-\frac{1}{2}, \frac{1}{2})$ *vertically upwards* $(y > 1/2)$ *curves rightward and intersects the* y *axis. Correspondingly, the same arch goes from the same intersection point and*

enters the saddle $(\frac{1}{2},\frac{1}{2})$. *Thus the separatrix leaving* $(-\frac{1}{2},\frac{1}{2})$ *for* $y>1/2$ *enters the other saddle* $(\frac{1}{2},\frac{1}{2})$. *Similarly, the separatrix leaving* $(\frac{1}{2},\frac{1}{2})$ *for* $y<1/2$ *enters the other saddle. Inside the separatrix loops a trajectory surrounding a center is closed because of the symmetry* $x\to -x$, $t\to -t$, $y\to y$, *so a center of the linearized system is a center of the nonlinear system of ODEs. The phase portrait is depicted in the figure. A different way to reach the same conclusion is to check that the ODEs are Hamilton equations* $\dot y=-\frac{\partial H}{\partial x}$, $\dot x=\frac{\partial H}{\partial y}$ *(because the off-diagonal terms of the Jacobian we found before are* $2y-1=-\frac{\partial^2 H}{\partial x\partial y}$ *and* $1-2y=\frac{\partial^2 H}{\partial x\partial y}$*), for the Hamiltonian* $H(x,y)=x^2y-\frac{x^2}{2}+\frac{y^3}{3}-\frac{y^2}{2}$. *The E.P. of Hamilton's equations can only be stable centers and unstable saddles.*

2. Solve

$$\begin{aligned}\frac{\partial^2 u}{\partial t^2}&=c^2\frac{\partial^2 u}{\partial x^2},\quad 0<x<L,\ t>0,\\ u(0,t)&=h(t),\quad u(L,t)=0,\ t>0,\\ u(x,0)&=\frac{\partial u}{\partial t}(x,0)=0,\quad 0<x<L.\end{aligned}$$

Solution. The eigenfunctions of the spatial part of the wave equation with Dirichlet boundary conditions are $\varphi_n(x)=\sin\frac{n\pi x}{L}$, $n=1,2,\dots$. *Let us expand the solution of the IVBP in series of eigenfunctions:* $u(x,t)=\sum_{n=1}^\infty u_n(t)\sin\frac{n\pi x}{L}$. *We now multiply the inhomogeneous wave equation by* $\varphi_n(x)$ *and integrate over* x *from 0 to* L. *The result is*

$$0=\frac{2}{L}\int_0^L\left(\frac{\partial^2 u}{\partial t^2}-c^2\frac{\partial^2 u}{\partial x^2}\right)\sin\frac{n\pi x}{L}dx=\ddot u_n-\frac{2c^2}{L}\left(\sin\frac{n\pi x}{L}\frac{\partial u}{\partial x}-\frac{n\pi}{L}u\cos\frac{n\pi x}{L}\right)\Big|_{x=0}^{L}$$
$$+\frac{n^2c^2\pi^2}{L^2}u_n=\ddot u_n+\frac{n^2c^2\pi^2}{L^2}u_n-\frac{2n\pi c^2}{L^2}h(t)\Longrightarrow \ddot u_n+\frac{n^2c^2\pi^2}{L^2}u_n=\frac{2n\pi c^2}{L^2}h(t).$$

This ODE is to be solved with initial conditions $u_n(0)=\dot u_n(0)=0$. *The solution of this IVP is*

$$u_n(t)=\frac{2c}{L}\int_0^t\sin\frac{n\pi c(t-s)}{L}h(s)\,ds,$$

as it follows from variation of parameters and can be checked by substituting in the ODE. Therefore the solution of the wave equation with inhomogeneous boundary condition and zero initial data is

$$u(x,t)=\frac{2c}{L}\sum_{n=1}^\infty\sin\frac{n\pi x}{L}\int_0^t\sin\frac{n\pi c(t-s)}{L}h(s)\,ds.$$

3. (a) Discretize the telegrapher's equation

$$\frac{\partial^2 u}{\partial t^2}+a\frac{\partial u}{\partial t}-b\,u=\frac{\partial^2 u}{\partial x^2},$$

by approximating all derivatives by centered differences. (b) Find the scheme CFL stability condition assuming that $|a|\,\Delta t$, $|b|(\Delta t)^2$ (Δt = time step) are small enough.

Solution. (a)

$$\Big(1+\frac{a\Delta t}{2}\Big)u_j^{n+1}+\Big(1-\frac{a\Delta t}{2}\Big)u_j^{n-1}-2u_j^n-s\,(u_{j+1}^n+u_{j-1}^n-2u_j^n)-b(\Delta t)^2u_j^n=0,$$

with $s=(\Delta t/\Delta x)^2$.

(b)The CFL condition is found by substituting $u_j^n=e^{ij\alpha}Q^n$ *in the difference equation. We find*

$$\Big(1+\frac{a\Delta t}{2}\Big)Q^2-2\Big[1-s(1-\cos\alpha)+\frac{b(\Delta t)^2}{2}\Big]Q+1-\frac{a\Delta t}{2}=0.$$

The solutions are complex conjugate and have modulus smaller than 1 provided: $0<|a|\,\Delta t<2$ *and*

$$\Big[1-s(1-\cos\alpha)+\frac{b(\Delta t)^2}{2}\Big]^2<\Big(1+\frac{a\Delta t}{2}\Big)\Big(1-\frac{a\Delta t}{2}\Big)=1-\Big(\frac{a\Delta t}{2}\Big)^2,$$

or, equivalently,

$$-\sqrt{1-\frac{a^2(\Delta t)^2}{4}}<1-2s\sin^2\frac{\alpha}{2}+\frac{b(\Delta t)^2}{2}<\sqrt{1-\frac{a^2(\Delta t)^2}{4}}.$$

This holds if

$$s\sin^2\frac{\alpha}{2}<\frac{1}{2}+\frac{b(\Delta t)^2}{4}+\frac{1}{2}\sqrt{1-\frac{a^2(\Delta t)^2}{4}}=1+\frac{b(\Delta t)^2}{4}-\frac{1}{2}\left(1-\sqrt{1-\frac{a^2(\Delta t)^2}{4}}\right),$$

i.e.,

$$s\sin^2\frac{\alpha}{2}<1+\frac{b(\Delta t)^2}{4}-\frac{a^2(\Delta t)^2}{8}\frac{1}{1+\sqrt{1-\frac{a^2(\Delta t)^2}{4}}}.$$

From this inequality and from $|sin(\alpha/2)|<1$ *and* $-1<-1/(1+B)$, *we find the CFL condition*

$$s<1-\frac{(\Delta t)^2(a^2-2b)}{8}\Leftrightarrow\frac{\Delta x}{\Delta t}>\frac{1}{\sqrt{1-(a^2-2b)\frac{(\Delta t)^2}{8}}}.$$

Here we must have $0<(a^2-2b)\frac{(\Delta t)^2}{8}<1$, *which places another restriction on the coefficients* a, b *and the time step* Δt.

4. (a) Find the values of L for which the boundary value problem (BVP):

$$\begin{aligned}&\frac{d^2u}{dx^2}+u=f(x),\quad 0<x<L,\\&u(0)=0=u(L).\end{aligned}$$

has a unique solution. (b) Find the solution(s) of the BVP whenever they exist.

Solution. Variation of parameters, $u(x) = A(x)\cos x + B(x)\sin x$ *produces the equations:*

$$A'\cos x + B'\sin x = 0, \quad -A'\sin x + B'\cos x = f(x) \Longrightarrow A' = -f(x)\sin x,\ B' = f(x)\cos x.$$

Then

$$u(x) = -\cos x \int_0^x f(s)\sin s\, ds + \sin x \left(a + \int_0^x f(s)\cos s\, ds\right) = a\sin x + \int_0^x \sin(x-s) f(s) ds,$$

solves the ODE with $u(0) = 0$. *The other boundary condition yields*

$$0 = a\sin L + \int_0^L \sin(L-s) f(s) ds.$$

This equation can be solved for a, *which gives the unique solution to the BVP,*

$$u(x) = \int_0^x f(s)\sin(x-s)\, ds - \frac{\sin x}{\sin L}\int_0^L \sin(L-s) f(s) ds,$$

provided $L \neq n\pi$ *with* $n = 1, 2, \ldots$. *For* $L = n\pi$, *the BVP has no solution unless*

$$\int_0^{n\pi} f(s)\sin s\, ds = 0.$$

In the latter case, the solution of the BVP is

$$u(x) = \int_0^x f(s)\sin(x-s)\, ds + c\sin x, \qquad \text{for arbitrary } c.$$

The same results can be found by the eigenfunction expansion $u(x) = \sum_{n=1}^{\infty} u_n \sin\frac{n\pi x}{L}$. *Then* $u_n = \frac{f_n}{1 - n^2\pi^2/L^2}$, $f_n = \int_0^L f(x)\sin\frac{n\pi x}{L} ds$, *which leads to the same conclusions.*

5. Solve the following Poisson equation $\nabla^2 u = r\sin(2\theta)$ inside the half-circle of unit radius, $0 < r < 1$, $0 < \theta < \pi$, with the boundary conditions $u(1,\theta) = 0$, and $u(r,0) = u(r,\pi) = 0$.

   Solution. We try a particular solution $U = (Ar^3 + Br^2)\sin(2\theta)$. *Then* $6A + 3A - 4A = 1$ *for any* B, *and* $A = 1/5$. $U(1,\theta) = (B + 1/5)\sin(2\theta) = 0$ *for* $B = -1/5$. *Furthermore,* $U(r,0) = U(r,\pi/2) = 0$. *Thus our particular solution satisfies all three boundary conditions and is the sought solution of the BVP:*

$$u(r,\theta) = \frac{r^2(r-1)}{5}\sin(2\theta).$$

In a more systematic way, the eigenfunctions for the angular part of the solution are $\sin(n\theta)$, *and insertion of* $u(r,\theta) = \sum_{n=1}^{\infty} u_n(r)\sin(n\theta)$ *into the equation gives*

$$u_n'' + \frac{1}{r}u_n' - \frac{n^2}{r^2}u_n = r\delta_{n2} \Longrightarrow r^2 u_n'' + r u_n' - n^2 u_n = r^3\delta_{n2}.$$

Boundedness at the origin produces $u_n = a_n r^n + \frac{r^3}{5}\delta_{n2}$ *and the boundary condition at* $r = 1$ *yields* $a_n = -\frac{\delta_{n2}}{5}$, *thereby obtaining the same solution as the previous method.*