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Select and solve four out of the following five problems.

1. Given Frommer's system of ODEs (invariant under $x \to -x$, $t \to -t$, $y \to y$):

$$\frac{dy}{dt} = x - 2xy, \quad \frac{dx}{dt} = -y + x^2 + y^2,$$

find, classify and ascertain the stability of its equilibrium points (constant solutions) in the phase plane (x, y).

2. Solve

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= c^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L,\ t > 0, \\ u(0,t) &= h(t), \quad u(L,t) = 0,\ t > 0, \\ u(x,0) &= \frac{\partial u}{\partial t}(x,0) = 0, \quad 0 < x < L. \end{aligned}$$

3. (a) Discretize the telegrapher's equation

$$\frac{\partial^2 u}{\partial t^2} + a\frac{\partial u}{\partial t} - b\,u = \frac{\partial^2 u}{\partial x^2},$$

by approximating all derivatives by centered differences. (b) Find the scheme CFL stability condition assuming that $|a|\,\Delta t$, $|b|(\Delta t)^2$ (Δt = time step) are small enough.

4. (a) Find the values of L for which the boundary value problem (BVP):

$$\begin{aligned} \frac{d^2 u}{dx^2} + u &= f(x), \quad 0 < x < L, \\ u(0) &= 0 = u(L). \end{aligned}$$

has a unique solution. (b) Find the solution(s) of the BVP whenever they exist.

5. Solve the following Poisson equation $\nabla^2 u = r\sin(2\theta)$ inside the half-circle of unit radius, $0 < r < 1$, $0 < \theta < \pi$, with the boundary conditions $u(1,\theta) = 0$, and $u(r,0) = u(r,\pi) = 0$.