UC3M Final, January 20, 2016

Advanced Mathematical Methods

Name: _____.

Exercise 1 (2 points)

The dynamics for an empty, homogeneous and isotropic universe reduces to the following differential equation: the square of the rate of change of the size of the Universe is proportional to the square of the size of the Universe.

- Obtain the differential equation and identify the dependent, independent variable and the parameter(s). (0.75 point)
- Solve the differential equation and obtain the evolution of the size of the Universe assuming that we know that it is expanding. (0.75 point)
- Suppose that at some initial time $t_0 = 0$ we have a certain size. At what time is the size twice the initial size? Express your answer in terms of the parameter(s). (0.5 point)

Solution:

- The differential equation is (0.5 point)

$$\dot{a}^2 = ka^2,$$

where a is the size and the dependent variable, t is time and the independent variable and k the proportionality factor and a parameter which has to be positive since we have a square on the left hand side (0.25 point).

- Taking the square we obtain

$$\dot{a} = \pm\sqrt{k}a$$

(0.25 point). We have to take the positive sign since we know the Universe is expanding (0.25 point). Integrating we obtain

$$a = a(t_0)e^{\sqrt{k}(t-t_0)}$$

(0.25 point).

- Transforming we obtain

$$t = t_0 + \frac{1}{\sqrt{k}} \ln \frac{a(t)}{a(t_0)}$$

(0.25 point). Finally we have $t_0 = 0$, $\frac{a(t)}{a(t_0)} = 2$, thus we have

$$t = \frac{1}{\sqrt{k}} \ln 2$$

(0.25 point).

Exercise 2 (2 points)

Consider the system

$$x' = 2y(1 + y^2),$$
$$y' = -2x(1 + x^2).$$

- Obtain the equilibrium point(s) and consider the linearisation there (0.5 point).

- Obtain the (real) solution to the linearised system. (0.75 point)
- Classify the equilibrium point and make a rough sketch of the behaviour of the solutions to the linearised system near the equilibrium point. (0.5 point)
- What can we say about the non-linear system from the linearised system? (0.25 point)

Sol.:

- Setting to zero the derivatives, we see that only $(x, y) = (0, 0)$ is an equilibrium point (0.25 point) Taking the Jacobian we see that the linearisation there is

$$x' = 2y,$$
$$y' = -2x$$

  (0.25 point).
- The eigenvalues are $\lambda = \pm 2i$. The eigenvector corresponding to $\lambda = 2i$ is $(1, i)$ (0.25 point) . The solutions are of the form

$$e^{2it}(1, i)$$

  (0.25 point) and separating real and imaginary part we finally obtain

$$x = C_1 \cos 2t + C_2 \sin 2t,$$
$$y = -C_1 \sin 2t + C_2 \cos 2t.$$

  (0.25 point)
- The equilibrium point is a center (0.167 point) and thus the trajectories are ellipses around the origin (0.167 point). When we look at the direction field at $(1, 0)$ we obtain $(0, -2)$, which means that the trajectories run clockwise (0.167 point).
- There is no real part in the eigenvalue, thus we cannot apply the theorem we saw in class (0.25 point). We did not see this in class, but one can actually deduce that it will be either a spiral sink, a spiral source or a center (0.25 point extra if know this).

Exercise 3 (2 points)

Give the general solution in any interval outside $x = 0$ of

$$x^2 y'' - 3xy' + 4y = 0$$

Hint: Try $y = x^\lambda$ and determine λ. Obtain a second solution using the method of variation of parameters. Show that the two solutions are linearly independent and give the general solution to the equation.

Sol.: Substituing $y = x^\lambda$ we obtain $\lambda = 2$ (0.25 points). To find another solution we try

$$y = x^2 u(x),$$

where u is a function of x (0.25 point). Then

$$y' = 2xu + x^2 u', y'' = 2u + 4xu' + x^2 u''.$$

Substituing we arrive at

$$xu'' + u' = 0$$

(0.25 point) Introduce $u' = v$, then we have that

$$xv' = -v$$

(0.25 point). As a result we have

$$v = Cx^{-1}$$

(0.25 point) and

$$u = C_1 \ln x + C_2.$$

Thus the other solution if we drop C_2 since it gives us just the same thing is $x^2 \ln x$ (0.25 point). Both solutions are independent since the Wronskian is x^3 which is always different from zero for $x \neq 0$ (0.25 point). The general solution to the problem is thus

$$y = x^2(C_1 + C_2 \ln x)$$

(0.25 point).

Exercise 4 (1point + 1point)

- Is the following linear PDE for $u(x, y, z)$ hyperbolic, parabolic or elliptic? Provide a reason.

$$u_{yy} - u_{zz} + 3u_x - 5u_y + 6u_z + u = 0.$$

Solution The only thing that matters are the derivatives of higher order and thus the diagonal matrix $(0, 1, -1)$ (0.5 point). It does not fulfil any of the criteria, so it is neither hyperbolic, parabolic or elliptic (0,5 point).

- For solutions on the interval $0 < x < 1$ of

$$[p(x)y']' - q(x)y + \lambda r(x)y = 0, \tag{1}$$

with p, q real ith boundary conditions

$$y(0) + y'(0) = 0, \tag{2}$$
$$y(1) + y'(1) = 0. \tag{3}$$

show **Lagrange's identity**

$$(\mathcal{L}[u], v) - (u, \mathcal{L}[v]) = 0.$$

where

$$\mathcal{L}[y] = -[p(x)y']' + q(x)y,$$

and $(,)$ is the **inner product**

$$(u, v) = \int_0^1 u(x)v(x)dx.$$

Sol.: It is a straightforward computation using integration by parts (1 point).

Exercise 5 (2 points)

Find the solution to heat conduction problem

$$u_t = u_{xx} + xe^{-t},$$

with BC

$$u(0, t) = 0,$$
$$u_x(1, t) + u(1, t) = 0.$$

and initial condition

$$u(x,0) = 0.$$

We do the separation of variables $u = X(x)T(t)$ (0.25 points) and obtain for the associated homogeneous equation

$$T'X = X''T.$$

Divide by $u = XY$ and separate into parts which have to be constants:

$$\frac{T'}{T} = \frac{X''}{X} = -\lambda.$$

Thus we have

$$T' + \lambda T = 0,$$
$$X'' + \lambda X = 0.$$

The conditions on the boundary translate into condition for X

$$X(0) = 0,$$
$$X'(1) + X(1) = 0.$$

(0.25 point) As usual we consider the different possibilities for λ. Suppose that $\lambda = 0$. Then

$$X = C_1 x + C_2.$$

The first condition implies that $C_2 = 0$. The second that $C_1 = 0$ (0.25 point). Consider the case that $\lambda < 0$. Then the solution is

$$y = C_1 e^{\sqrt{-\lambda}x} + C_2 e^{-\sqrt{-\lambda}x}.$$

The first BC implies

$$C_1 + C_2 = 0,$$

which means that $C_1 = -C_2$ and the second

$$\sqrt{-\lambda}C_1 e^{\sqrt{-\lambda}} - \sqrt{-\lambda}C_2 e^{-\sqrt{-\lambda}} + C_1 e^{\sqrt{-\lambda}} + C_2 e^{-\sqrt{-\lambda}} = 0.$$

Putting the first condition and dividing by C_1

$$\sqrt{-\lambda}(e^{\sqrt{-\lambda}} + e^{-\sqrt{-\lambda}}) + e^{\sqrt{-\lambda}} - e^{-\sqrt{-\lambda}} = 0,$$

or

$$\sqrt{-\lambda} = -\tanh\sqrt{\lambda},$$

which has no solutions since tanh is always positive for a positive entry and $\lambda \neq 0$ for this case (0.25 points).

Consider the case that $\lambda > 0$ then the solution is

$$y = C_1 \cos\sqrt{\lambda}x + C_2 \sin\sqrt{\lambda}x.$$

The first condition $X(0) = 0$ implies that $C_1 = 0$. The second condition gives us

$$C_2(\sqrt{\lambda}\cos\sqrt{\lambda} + \sin\sqrt{\lambda}) = 0 \tag{4}$$

Suppose that $\cos\sqrt{\lambda}=0$, then we know that $\sin\sqrt{\lambda}\neq 0$ and (4) wouldn't be satisfied. As a consequence we can assume that $\cos\sqrt{\lambda}\neq 0$. Equation (4) cannot be solved analytically but we one solve it numerically or to have an idea graphically. Since we can divide by $\cos\sqrt{\lambda}$ and we know then that λ has to satisfy

$$\sqrt{\lambda}=-\tan\sqrt{\lambda}.$$

The eigenvalues can be computed and the first three are 4,116, 24,14 and 63,66. The numerical values are not important for as and let us call them λ_n as usual (0.25 point). The eigenfunctions are thus of the form $\sin\sqrt{\lambda}x$. For the case $\lambda>0$ we still need to normalise. Call the factors c_n which means that

$$1=\int_0^1\phi_n^2dx=c_n^2\int_0^1\sin^2\sqrt{\lambda_n}xdx=\frac{1}{2}c_n^2\int_0^1(1-\cos 2\sqrt{\lambda_n}x)dx$$
$$=\frac{1}{2}c_n^2(1-\frac{\sqrt{\lambda_n}}{2}\sin 2\sqrt{\lambda_n})=\frac{1}{2}c_n^2(1-\sqrt{\lambda}\sin\sqrt{\lambda_n}\cos\sqrt{\lambda_n})=\frac{1}{2}c_n^2(1+\cos^2\sqrt{\lambda_n}),$$

where in the last step we have used (4). Thus we have to choose

$$c_n=\frac{\sqrt{2}}{\sqrt{1+\cos^2\sqrt{\lambda_n}}},$$

and our normalised eigenfunctions are

$$\frac{\sqrt{2}\sin\sqrt{\lambda_n}x}{\sqrt{1+\cos^2\sqrt{\lambda_n}}}$$

(0.25 point) Coming back to our original problem we will look for a solution of the form

$$u=\sum_{n=1}^{\infty}b_nc_n\sin\sqrt{\lambda_n}x.$$

We need to express xe^{-t} in terms of these eigenfunctions

$$\gamma_n=\int_0^1e^{-t}xc_n\sin\sqrt{\lambda_n}x=e^{-t}c_n\int_0^1x\sin\sqrt{\lambda_n}x=e^{-t}c_n[-\frac{x}{\sqrt{\lambda_n}}\cos\sqrt{\lambda_n}x+\frac{1}{\lambda_n}\sin\sqrt{\lambda_n}x]|_0^1$$
$$e^{-t}c_n[-\frac{1}{\sqrt{\lambda_n}}\cos\sqrt{\lambda_n}+\frac{1}{\lambda_n}\sin\sqrt{\lambda_n}]e^{-t}c_n[\frac{2}{\lambda_n}\sin\sqrt{\lambda_n}]=e^{-t}\frac{2\sqrt{2}\sin\sqrt{\lambda_n}}{\lambda_n\sqrt{1+\cos^2\sqrt{\lambda_n}}}.$$

(0.25 point) Since $u(x,0)=0$ we have that

$$b_n(0)=\int_0^1r(x)f(x)dx=0$$

and

$$b_n(t)=\frac{2\sqrt{2}\sin\sqrt{\lambda_n}}{\lambda_n\sqrt{1+\cos^2\sqrt{\lambda_n}}}\int_0^te^{-\lambda_n(t-s)}e^{-s}ds=\frac{2\sqrt{2}\sin\sqrt{\lambda_n}}{\lambda_n\sqrt{1+\cos^2\sqrt{\lambda_n}}}e^{-\lambda_nt}\frac{1}{\lambda_n-1}e^{(\lambda_n-1)s}|_0^t$$
$$=\frac{2\sqrt{2}\sin\sqrt{\lambda_n}}{\lambda_n\sqrt{1+\cos^2\sqrt{\lambda_n}}}\frac{1}{\lambda_n-1}(e^{-t}-e^{-\lambda_nt}).$$

(0.25 point) Finally our solution is

$$\sum_{n=1}^{\infty}\frac{4\sin\sqrt{\lambda_n}\sin\sqrt{\lambda_n}x}{\lambda_n(1+\cos^2\sqrt{\lambda_n})}\frac{1}{\lambda_n-1}(e^{-t}-e^{-\lambda_nt}).$$

(0.25 point)

Exercise 6 (2 points)

Solve the differential equation

$$y'' + 2y = -1 + |1 - 2x|$$

with BC

$$y(0) = 0,\ y(1) = 0,$$

through an eigenfunction expansion. The solution to the associated homogeneous is

$$y'' + \lambda y = 0$$

is

$$\sum_{n=1}^{\infty} \sqrt{2} \sin n\pi x,$$

where $\lambda = n^2\pi^2$ (0.5 point). The value of λ will never be $\mu = 2$ so that we can determine the coefficients of the eigenfunction expansion via

$$b_n = \frac{c_n}{\lambda_n - \mu}$$

(0.25 point) We just need to compute

$$c_n = -\sqrt{2} \int_0^1 dx \sin n\pi x[-1 + |1 - 2x|].$$

The first integral is

$$I_1 = \int_0^1 dx - \sin n\pi x = \frac{1}{n\pi} \cos n\pi x|_0^1 = \frac{1}{n\pi}[(-1)^n - 1]$$

(0.25 point) The second we do dividing the integral into two parts

$$I_2 = \int_0^{\frac{1}{2}} dx \sin n\pi x(1 - 2x),$$

and

$$I_3 = \int_{\frac{1}{2}}^1 dx \sin n\pi x(2x - 1).$$

We compute

$$I_2 = \frac{-1}{n\pi} \cos n\pi x|_0^{\frac{1}{2}} + \frac{2x}{n\pi} \cos n\pi x|_0^{\frac{1}{2}} - \frac{2}{n\pi} \int_0^{\frac{1}{2}} \cos n\pi x$$

$$= \frac{-1}{n\pi}(\cos \frac{n\pi}{2} - 1) + \frac{1}{n\pi} \cos \frac{n\pi}{2} - \frac{2}{n^2\pi^2} \sin \frac{n\pi}{2} = \frac{1}{n\pi} - \frac{2 \sin \frac{n\pi}{2}}{n^2\pi^2},$$

(0.25 point) and

$$I_3 = -\frac{1}{n\pi}(-1)^n - \frac{2 \sin \frac{n\pi}{2}}{n^2\pi^2}$$

(0.25 point). Thus the solution is

$$\phi = \sum_{n=1}^{\infty} \frac{8 \sin \frac{n\pi}{2} \sin n\pi x}{n^2\pi^2(n^2\pi^2 - 2)}$$

(0.25 point).