AEROSPACE ENGINEERING.
Advanced Mathematics. Final exam. January 23rd, 2014.
Computers, calculators, phones, tablets not allowed

Select and solve four out of the following five problems.

1. A string $(0, L)$ consists of two homogeneous parts of equal cross sections, smoothly joined at the point $x = x_0$, and having velocities, Young moduli (= tensions) and mass densities c_1, E_1, ρ_1 and c_2, E_2 and ρ_2, respectively, with $E_2 > E_1$, $\rho_2 > \rho_1$, $c_i^2 = E_i/\rho_i$. Find the displacement in such a string if both ends are fixed, $u(x,0) = x/x_0$ for $0 < x < x_0$, $u(x,0) = (L-x)/(L-x_0)$ for $x_0 < x < L$, and $u_t(x,0) = \partial u(x,0)/\partial t = 0$. *Hint.* u *and* $E(x)u_x$ *are continuous at* $x = x_0$ *with* $E(x) = E_1$ *for* $x < x_0$ *and* $E(x) = E_2$ *for* $x > x_0$.

Solution. We have to solve the IBVP:

$$\begin{aligned}
\rho_1 \partial_t^2 u = E_1 \partial_x^2 u, \quad 0 < x < x_0, \quad t > 0,\\
\rho_2 \partial_t^2 u = E_2 \partial_x^2 u, \quad x_0 < x < L, \quad t > 0,\\
u(x_0-,t) = u(x_0+,t), \quad E_1 \partial_x u(x_0-,t) = E_2 \partial_x u(x_0+,t),\\
u(0,t) = u(L,t) = 0,\\
u(x,0) = \begin{cases} \frac{x}{x_0}, & 0 < x < x_0,\\ \frac{L-x}{L-x_0}, & x_0 < x < L, \end{cases} \qquad \partial_t u(x,0) = 0.
\end{aligned}$$

The wave equation can be written as

$$\partial_t^2 u = [c(x)]^2 \partial_x^2 u, \quad c(x) = \begin{cases} c_1, & 0 < x < x_0,\\ c_2, & x_0 < x < L, \end{cases}$$

where $c_i^2 = E_i/\rho_i$. *By separation of variables,* $u(x,t) = X(x)T(t)$, *we get*

$$\frac{T''(t)}{T(t)} = [c(x)]^2 \frac{X''(x)}{X(x)} = -\lambda.$$

Using now $\lambda = \omega^2$, *we obtain:*

$$\begin{aligned}
&T'' + \omega^2 T = 0, t > 0,\\
&X'' + \frac{\omega^2}{c_1^2} X = 0, \quad 0 < x < x_0,
\end{aligned}$$

$$
\begin{aligned}
&X'' + \frac{\omega^2}{c_2^2} X = 0, \quad x_0 < x < L, \\
&X(x_0-) = X(x_0+), \quad E_1 X'(x_0-) = E_2 X'(x_0+), \\
&X(0) = 0 = X(L).
\end{aligned}
$$

The BVP is a Sturm-Liouville eigenvalue problem with eigenvalue ω^2 and weight function $1/c_1^2$ for $0 < x < x_0$ and $1/c_2^2$ for $x_0 < x < L$. The solution of the BVP is

$$
X(x) = \begin{cases} a \sin k_1 x, & 0 < x < x_0, \\ b \sin[k_2(x-L)], & x_0 < x < L, \end{cases}
$$

with $k_i = \omega/c_i$, $c_i = \sqrt{E_i/\rho_i}$. Continuity of X and $E_i X'$ at $x = x_0$ yields

$$
\begin{aligned}
a \sin k_1 x_0 &= b \sin[k_2(x_0 - L)], \\
a E_1 k_1 \cos k_1 x_0 &= b E_2 k_2 \cos[k_2(x_0 - L)],
\end{aligned}
$$

which produces the eigenvalue problem

$$
\tan \frac{\omega}{c_1} x_0 = \sqrt{\frac{E_1 \rho_1}{E_2 \rho_2}} \tan[\frac{\omega}{c_2}(x_0 - L)], \quad \frac{E_1 k_1}{E_2 k_2} = \sqrt{\frac{E_1 \rho_1}{E_2 \rho_2}}.
$$

There are infinitely many solutions $\omega_1 < \omega_2 < \ldots$. To each it corresponds the eigenfunction

$$
X_n(x) = \begin{cases} \frac{\sin \frac{\omega_n}{c_1} x}{\sin \frac{\omega_n}{c_1} x_0}, & 0 < x < x_0, \\ \frac{\sin\left[\frac{\omega_n}{c_2}(x-L)\right]}{\sin\left[\frac{\omega_n}{c_2}(x_0-L)\right]}, & x_0 < x < L. \end{cases}
$$

We have

$$
\begin{aligned}
\|X_n\|^2 &= \frac{\omega_n^2}{c_1^2} \int_0^{x_0} \sin k_1 x_0 \frac{\sin^2 \frac{\omega_n}{c_1} x}{\sin^2 \frac{\omega_n}{c_1} x_0} dx + \frac{\omega_n^2}{c_2^2} \int_{x_0}^{L} \frac{\sin^2\left[\frac{\omega_n}{c_2}(x-L)\right]}{\sin^2\left[\frac{\omega_n}{c_2}(x_0-L)\right]} dx \\
&= \frac{\omega_n^2 x_0}{2c_1^2 \sin^2 \frac{\omega_n}{c_1} x_0} + \frac{\omega_n^2 (L - x_0)}{2c_2^2 \sin^2\left[\frac{\omega_n}{c_2}(x_0 - L)\right]}.
\end{aligned}
$$

The solution that satisfies the initial conditions is

$$
\begin{aligned}
u(x,t) &= \sum_{n=1}^{\infty} a_n X_n(x) \cos \omega_n t, \\
a_n &= \frac{1}{c_1^2 \|X_n\|^2} \int_0^{x_0} u(x,0) X_n(x) dx + \frac{1}{c_2^2 \|X_n\|^2} \int_{x_0}^{L} u(x,0) X_n(x) dx \\
&= \frac{1}{\|X_n\|^2} \left(\frac{E_1}{x_0} + \frac{E_2}{L - x_0} \right).
\end{aligned}
$$

2. Consider the first order hyperbolic PDE $u_t + cu_x = x$ for $u(x,t)$, where subscripts mean partial derivatives. Approximate u_t and u_x by backward finite differences and by centered finite differences, respectively. Calculate the truncation error and the stability of the resulting numerical scheme.

Solution. The numerical scheme is

$$\frac{u_j^{(m)} - u_j^{(m-1)}}{\Delta t} + \frac{c}{2\Delta x}(u_{j+1}^{(m)} - u_{j-1}^{(m)}) = j\Delta x.$$

The truncation error is $\tau_j^{(m)}(\Delta t, \Delta x) = \frac{u_j^{(m)} - u_j^{(m-1)}}{\Delta t} + \frac{c}{2\Delta x}(u_{j+1}^{(m)} - u_{j-1}^{(m)}) - j\Delta x$, *where* $u_j^{(m)}$ *should be replaced by* $u(j(\Delta x), m(\Delta t))$ *and* $x = j(\Delta x)$, $t = m(\Delta t)$. *We find*

$$\tau_j^{(m)}(\Delta t, \Delta x) = \frac{u(x,t) - u(x,t-\Delta t)}{\Delta t} + \frac{c}{2\Delta x}[u(x+\Delta x, t) - u(x-\Delta x, t)] - x$$
$$= u_t + \frac{\Delta t}{2}u_{tt} + O((\Delta t)^2) + cu_x + \frac{c(\Delta x)^2}{6}u_{xxx} + O((\Delta x)^4) - x = O(\Delta t) + O((\Delta x)^2),$$

after using the PDE $u_t + cu_x = x$. *To ascertain the stability of the scheme, we substitute in it* $u_j^{(m)} = Q^m e^{ij\alpha\Delta x}$ *and keep the terms corresponding to the highest-order differences, thereby getting*

$$1 - \frac{1}{Q} + is\sin(\alpha\Delta x) = 0, \quad s = \frac{c\Delta t}{\Delta x}.$$

We find $Q^{-1} = 1 + is\sin(\alpha\Delta x)$, *and therefore*

$$|Q|^{-2} = 1 + s^2\sin^2(\alpha\Delta x) \geq 1 \Longrightarrow |Q| \leq 1,$$

and the scheme is stable for any real value of s.

3. Solve the wave equation $u_{tt} = c^2 u_{xx}$ on $x > 0$, $t > 0$, subject to the boundary condition $u_x(0,t) + hu(0,t) = q(t)$, $t > 0$, with $q(0) = 0$, and initial conditions $u(x,0) = u_t(x,0) = 0$ for $x > 0$.

Solution. Clearly $u = 0$ *for* $x > ct$ *due to the initial conditions. Then* $u = F(x-ct) + G(x+ct)$ *gives* $F(x) = G(x) = 0$ *for* $x > 0$. *For* $x < ct$, $u = F(x-ct)$ *(with* $x - ct < 0$*) which is not zero. Inserting in the boundary condition, we obtain*

$$F'(-ct) + hF(-ct) = q(t) \Longrightarrow F'(\xi) + hF(\xi) = q(-\xi/c) \quad \text{for } \xi = -ct.$$

This is a first order ODE which can be solved easily multiplying it by the integrating factor $e^{h\xi}$. *The solution is*

$$F(\xi) = \int_0^\xi e^{-h(\xi-s)}q(-s/c)\,ds = -c\int_0^{-\xi/c} e^{-h(\xi+c\tau)}q(\tau)d\tau,$$

and

$$u(x,t) = F(x-ct) = -c\int_0^{t-x/c} e^{-h(x-ct+c\tau)}q(\tau)d\tau, \quad x < ct.$$

4. Solve the exterior Dirichlet problem $\nabla^2 u = 0$ (in polar coordinates) on $1 < r < \infty$ for the boundary conditions $u(1,\theta) = 1$ for $0 \le \theta < \pi$, $u(1,\theta) = 0$ for $\pi \le \theta < 2\pi$, $|u(\infty,\theta)| < \infty$.

Solution. Separation of variables for $u_{rr} + r^{-1}u_r + r^{-2}u_{\theta\theta} = 0$ *produces* $r^2R'' + rR' - \lambda R = 0$ *and* $\Theta'' + \lambda\Theta = 0$ *if* $u = R(r)\Theta(\theta)$. *As usual* $\lambda = n^2$ *for eigenfunctions* $\cos n\theta$ *and* $\sin n\theta$. *The solutions of the corresponding Euler equations for* R *are* $R_n = a_n r^n + b_n/r^n$ *and* $R_0 = a_0 \ln r + b_0$. *Clearly solutions that are bounded at* $r = \infty$ *correspond to having* $a_n = 0$ *for all* $n \ge 0$. *Then the superposition principle yields*

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} \frac{1}{r^n}(A_n \cos n\theta + B_n \sin n\theta).$$

The boundary condition yields

$$\begin{aligned} A_0 &= \frac{1}{2\pi}\int_0^{2\pi} u(1,\theta)d\theta = \frac{1}{2}, \\ A_n &= \frac{1}{\pi}\int_0^{2\pi} u(1,\theta)\cos n\theta\, d\theta = \frac{1}{\pi}\int_0^{\pi} \cos n\theta\, d\theta = 0, \\ B_n &= \frac{1}{\pi}\int_0^{2\pi} u(1,\theta)\sin n\theta\, d\theta = \frac{1}{\pi}\int_0^{\pi} \sin n\theta\, d\theta = \frac{1-(-1)^n}{n\pi}. \end{aligned}$$

Then

$$u(r,\theta) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{(2n-1)\pi r^{2n-1}} \sin[(2n-1)\theta].$$

5. Solve Laplace's equation on the triangular region $\{(x,y), 0 < x < 1, 0 < y < x\}$ with boundary conditions $u(x,0) = x$ for $0 < x < 1$, $u(1,y) = 0$ for $0 < y < 1$ and $u(x,x) = 0$ for $0 < x < 1$. *Hint: Show that the solution of this problem is the same as the solution of Laplace's equation on the unit square with additional boundary conditions* $u(x,1) = 0$ *for* $0 < x < 1$ *and* $u(0,y) = -y$ *for* $0 < y < 1$ *(antisymmetric data with respect to the diagonal* $y = x$*).*

Solution. According to the hint, the solution on the square is antisymmetric with respect to the diagonal $y = x$, *so that* $u(y,x) = -u(x,y)$ *and therefore* $u(x,x) = 0$. *The solution on the unit square is split into two separate problems:* $u = u^I(x,y) + u^{II}(x,y)$, *where both these functions solve Laplace's equation with different boundary conditions: (i)* $u^I = 0$ *on* $x = 0,\ 1$ *and* $y = 1$, *and* $u^I = x$ *on* $y = 0$*; (ii)* $u^{II} = 0$ *on* $y = 0,\ 1$ *and* $x = 1$, *and* $u^{II} = -y$ *on* $x = 0$. *The solution of problem (i) is* $u^I = \sum_{n=1}^{\infty} a_n(y)\sin n\pi x$, *which inserted into Laplace's equation produces* $a_n'' - n^2\pi^2 a_n = 0$ *with* $a_n(1) = 0$. *The solution of this problem is* $a_n(y) = A_n \sinh[n\pi(y-1)]$. *The leftover boundary condition yields* $x = u^I(x,0) = -\sum_{n=1}^{\infty} A_n \sinh(n\pi)\sin n\pi x$. *Integration by parts produces*

$$\begin{aligned} A_n &= -\frac{2}{\sinh n\pi}\int_0^1 x\sin n\pi x\, dx = \frac{2x}{n\pi\sinh n\pi}\cos n\pi x|_0^1 - \frac{2}{n\pi\sinh n\pi}\int_0^1 \cos n\pi x\, dx \\ &= (-1)^n \frac{2}{n\pi\sinh n\pi}. \end{aligned}$$

Then

$$u^I(x,y) = \sum_{n=1}^{\infty} (-1)^n \frac{2}{n\pi \sinh n\pi} \sin n\pi x \sinh[n\pi(y-1)].$$

Similarly $u^{II}(x,y) = -u^I(y,x)$*:*

$$u^{II}(x,y) = -\sum_{n=1}^{\infty} (-1)^n \frac{2}{n\pi \sinh n\pi} \sin n\pi y \sinh[n\pi(x-1)],$$

and therefore the solution of the BVP is

$$u(x,y) = \sum_{n=1}^{\infty} \frac{2(-1)^n}{n\pi \sinh n\pi} \{\sin(n\pi x) \sinh[n\pi(y-1)] - \sin(n\pi y) \sinh[n\pi(x-1)]\}.$$