AEROSPACE ENGINEERING.
Advanced Mathematics. Final exam. January 22nd, 2014.

Select and solve four out of the following five problems.

1. A sultan and his vizier order coffee and receive identical cups at the same temperature at the same time. The sultan adds a small amount of cool cream immediately, but does not drink his coffee until 10 minutes later. The vizier waits 10 minutes, and then adds the same amount of cool cream and begins to drink. Who drinks the hotter coffee? Explain and reason your answer. *Hint: Use Newton's cooling law and energy conservation.*

   Solution. Let c and c' be the specific heats of coffee and cream, respectively. If we add a mass m' of cream (at temperature T') to that m of coffee (at T), the final temperature satisfies $(mc+m'c')T_f = mcT + m'c'T'$, whereby

$$T_f = \frac{mcT + m'c'T'}{mc+m'c'} \approx T + \frac{m'c'}{mc}T',$$

   provided $m'c' \ll mc$. Now we use Newton's cooling law $\frac{dT}{dt} = -h(T-T_r)$, where T_r is the room temperature. The sultan initial condition is $T(0) = \frac{mcT_0}{mc+m'c'} + \frac{m'c'T'}{mc+m'c'}$, where T_0 is the initial temperature of the coffee. After 10 minutes we have $T_s = T_r + (\frac{mcT_0+m'c'T'}{mc+m'c'} - T_r)e^{-10h}$. The vizier's cup is at temperature $T_{10} = T_r + (T_0 - T_r)e^{-10h}$ after 10 minutes and $T_v = \frac{mcT_{10}+m'c'T'}{mc+m'c'}$ after he pours cream. We have

$$T_v - T_s = \frac{m'c'}{mc+m'c'}(T'-T_r)(1-e^{-10h}) \approx \frac{m'c'}{mc}(T'-T_r)(1-e^{-10h}) > 0$$

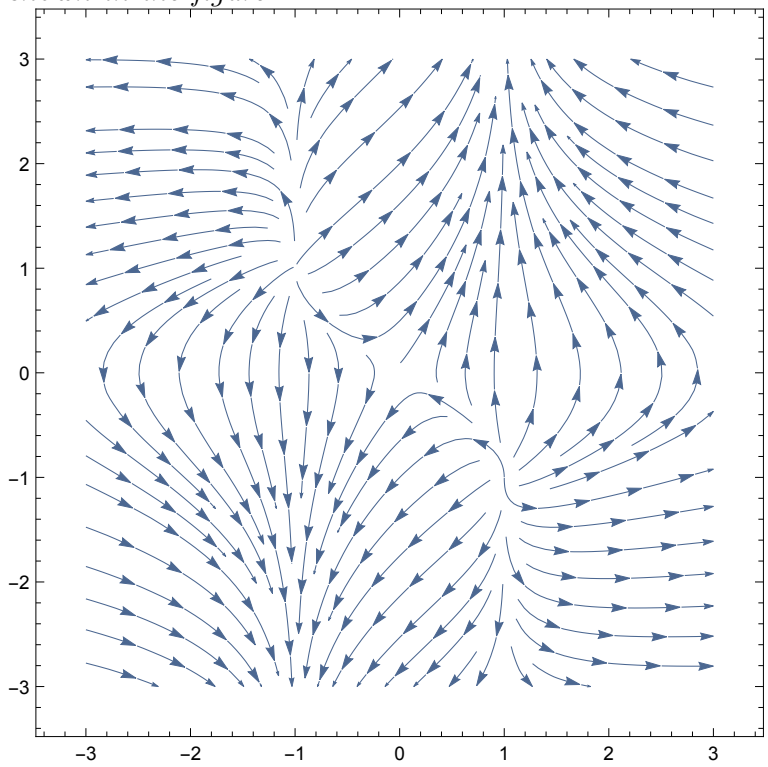
   (if $T' > T_r$), and therefore the vizier drinks his coffee warmer than the sultan does his.

2. Find all equilibrium points for the system $x' = y(1-x^2)$, $y' = x+y$, linearize it about them, classify the EPs, and calculate the corresponding eigenvectors in the cases of real eigenvalues. Use this information to sketch the phase plane.

   Solution. The EP satisfy $y(1-x^2) = 0$ so that either $y = 0$ or $x = \pm 1$. Since $x + y = 0$, the first possibility leads to $x = 0$ and the second to $y = \mp 1$. Therefore, the EP are $(0,0)$, $(1,-1)$ and $(-1,1)$. Linearization about these EPs gives the Jacobian matrices

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix},$$

for the EP at the origin and those at $(\pm 1, \mp 1)$*, respectively. The first EP is a saddle point: the eigenvalues solve* $\lambda^2 - \lambda - 1 = 0$*, therefore* $\lambda = (1 \pm \sqrt{5})/2$*. The other EPs are unstable nodes: the eigenvalues solve* $\lambda^2 - 3\lambda + 2 = 0$*, and therefore are 2 and 1. The eigenvectors for* $(0,0)$ *are* $(2, 1+\sqrt{5})$ *for* $(1+\sqrt{5})/2$ *and* $(2, 1-\sqrt{5})$ *for* $(1-\sqrt{5})/2$*. The eigenvectors for* $(\pm 1, \mp 1)$ *are* $(1,1)$ *for* 2 *(ordinary eigendirection) and* $(0,1)$ *for* 1 *(extraordinary eigendirection). Knowing all eigenvectors and the type of EP, we can sketch the phase portrait, at least near the EP. The phase portrait is shown in the figure.*



3. Solve the following IBVP:

$$\begin{aligned} &\frac{\partial u}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L,\, t > 0, \\ &u(x,0) = 0, \quad 0 < x < L, \\ &u(0,t) = \sin t, \quad u(L,t) = 0, \quad t > 0. \end{aligned}$$

Solution. Let us split $u = (1 - \frac{x}{L})\sin t + v(x,t)$*. Then* v *solves the IBVP:*

$$\begin{aligned} &\frac{\partial v}{\partial t} - k\,\frac{\partial^2 v}{\partial x^2} = -\left(1 - \frac{x}{L}\right)\cos t, \quad 0 < x < L,\, t > 0, \\ &v(x,0) = 0, \quad 0 < x < L, \\ &v(0,t) = 0, \quad v(L,t) = 0, \quad t > 0. \end{aligned}$$

Inserting the eigenfunction expansion $v(x,t) = \sum_{n=1}^{\infty} a_n(t) \sin \frac{n\pi x}{L}$ *in the equation, we obtain*

$$a_n' + k\frac{n^2\pi^2}{L^2} a_n = \frac{2}{L}\cos t \int_0^L \left(1 - \frac{x}{L}\right) \sin \frac{n\pi x}{L}\, dx = -\frac{2}{n\pi}\cos t,$$

to be solved with the initial condition $a_n(0) = 0$. *A particular solution is* $a_n = A_n \cos t + B_n \sin t$ *which, inserted in the ODE yields*

$$-A_n \sin t + B_n \cos t + k\frac{n^2\pi^2}{L^2}(A_n \cos t + B_n \sin t) = -\frac{2}{n\pi}\cos t.$$

Then we find

$$A_n = k\frac{n^2\pi^2}{L^2}B_n, \quad B_n + k\frac{n^2\pi^2}{L^2}A_n = -\frac{2}{n\pi}.$$

From this equation, we get

$$\left(1 + \frac{k^2n^4\pi^4}{L^4}\right)A_n = -\frac{2kn\pi}{L^2}, \quad \left(1 + \frac{k^2n^4\pi^4}{L^4}\right)B_n = -\frac{2}{n\pi},$$

and therefore

$$a_n = -\frac{2L^4}{n\pi(L^4 + k^2n^4\pi^4)}\sin t - \frac{2kn\pi L^2}{L^4 + k^2n^4\pi^4}(\cos t - e^{-kn^2\pi^2t/L^2}).$$

The solution of the original IBVP is therefore

$$\begin{aligned} u(x,t) &= \left(1 - \frac{x}{L} - \frac{2L^4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(L^4 + k^2n^4\pi^4)n}\sin\frac{n\pi x}{L}\right)\sin t \\ &- 2\pi kL^2\sum_{n=1}^{\infty}\frac{n}{L^4 + k^2n^4\pi^4}(\cos t - e^{-kn^2\pi^2t/L^2})\sin\frac{n\pi x}{L}. \end{aligned}$$

4. Consider the telegrapher's equation for the voltage $V(x,t)$ at point x of the line at time t (L, C, R and G are positive constants)

$$\frac{\partial^2 V}{\partial x^2} = LC\,\frac{\partial^2 V}{\partial t^2} + (RC + GL)\,\frac{\partial V}{\partial t} + GR\,V.$$

Solve the telegrapher's equation on $0 < x < l$, for $t > 0$ with the following boundary and initial conditions: $V(0,t) = 0$, $\frac{\partial V}{\partial x}(l,t) = 0$ and $V(x,0) = 1$ for $0 < x < \frac{l}{4}$, and $V(x,0) = 0$ for $\frac{l}{4} < x < l$, $\frac{\partial V}{\partial t}(x,0) = 0$. Simplify the solution for the case $\frac{R}{L} = \frac{G}{C}$.
Solution. The eigenvalue problem for the spatial part of the equation is

$$\begin{cases} X'' + \lambda X = 0,\ 0 < x < l, \\ X(0) = 0,\ X'(l) = 0. \end{cases}$$

Its solutions are $X_n(x) = \sin\frac{(2n-1)\pi x}{2l}$, $\lambda_n = \left(\frac{(2n-1)\pi}{2l}\right)^2$, *with* $n = 1, 2, \ldots$. *Inserting the eigenfunction expansion* $V(x,t) = \sum_{n=1}^{\infty} a_n(t)\sin\frac{2n-1)\pi x}{2l}$ *in the telegrapher's equation, we get*

$$\begin{cases} LCa_n'' + (RC + GL)\,a_n' + \left(GR + \frac{(2n-1)^2\pi^2}{4l^2}\right)a_n = 0, \\ a_n'(0) = 0,\ a_n(0) = \frac{2}{l}\int_0^{l/4}\sin\frac{(2n-1)\pi x}{2l}\,dx = \frac{4}{\pi(2n-1)}\left(1 - \cos\frac{(2n-1)\pi}{8}\right). \end{cases}$$

Now, let

$$\alpha = \frac{RC + GL}{2LC}, \quad \gamma_n = \frac{GR}{LC} + \frac{(2n-1)^2\pi^2}{4l^2LC}, \quad \omega_n = \sqrt{\gamma_n - \alpha^2}.$$

Then the general solution of the ODE is

$$a_n(t) = e^{-\alpha t}(A_n \cos \omega_n t + B_n \sin \omega_n t).$$

The initial conditions give $A_n = a_n(0)$, $B_n = \alpha A_n/\omega_n$. *Thus we obtain*

$$V(x,t) = -\frac{4}{\pi} e^{-\alpha t} \sum_{n=1}^{\infty} \frac{\omega_n \cos \omega_n t + \alpha \sin \omega_n t}{(2n-1)\omega_n} \left(1 - \cos \frac{(2n-1)\pi}{8}\right) \sin \frac{(2n-1)\pi x}{2l}.$$

If $\frac{R}{L} = \frac{G}{C}$, $\alpha = R/L$ *and* $\omega_n = (2n-1)\pi a/(2l)$, *where* $a = 1/\sqrt{LC}$. *In this case, we get*

$$V(x,t) = -\frac{4e^{-\frac{Rt}{L}}}{\pi^2 a} \sum_{n=1}^{\infty} \frac{(2n-1)\pi a \cos \frac{(2n-1)\pi a t}{2l} + \frac{2Rl}{L} \sin \frac{(2n-1)\pi a t}{2l}}{(2n-1)^2} \left(1 - \cos \frac{(2n-1)\pi}{8}\right) \sin \frac{(2n-1)\pi x}{2l}.$$

5. Consider the one dimensional wave equation

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0.$$

   (a) Approximate $\frac{\partial u}{\partial t}$ by a backward difference and $\frac{\partial u}{\partial x}$ by a forward difference, write the resulting finite difference numerical scheme and show that the scheme is consistent.

   (b) Use von Neumann's stability analysis to deduce a criterion for stability and, therefore, convergence of the scheme. How does it compare with the CFL condition for the usual finite difference scheme for the second order wave equation?

Solution. We use

$$\frac{\partial u}{\partial t} \approx \frac{u_j^n - u_j^{n-1}}{\Delta t}, \quad \frac{\partial u}{\partial x} \approx \frac{u_{j+1}^n - u_{j-1}^n}{\Delta x},$$

thereby obtaining the scheme

$$u_j^n + \sigma(u_{j+1}^n - u_j^n) = u_j^{n-1}, \quad \sigma = \frac{c\Delta t}{\Delta x}.$$

The truncation error,

$$\begin{aligned} \tau_j^n &= \frac{u(x,t) - u(x,t-\Delta t)}{\Delta t} + c\frac{u(x+\Delta x,t) - u(x,t)}{\Delta x} - \frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x} \\ &= -\frac{\Delta t}{2}\frac{\partial^2 u}{\partial t^2} + c\frac{\Delta x}{2}\frac{\partial^2 u}{\partial x^2}, \end{aligned}$$

tends to 0 as Δx, Δt *do, therefore the scheme is consistent. Let now* $u_j^n = Q^n e^{ikj}$, *where* $k = \alpha \Delta x$. *Inserting this in the numerical scheme and dividing the result by* u_j^n, *we obtain*

$$1 - \frac{1}{Q} + \sigma(e^{ik} - 1) = 0 \Longrightarrow Q = \frac{1}{1 - \sigma(e^{ik} - 1)}.$$

Then

$$|Q|^2 = \frac{1}{[1-(1-\cos k)\sigma]^2 + \sigma^2 \sin^2 k} = \frac{1}{1 - 2\sigma(1-\cos k) + \sigma^2(1 + \cos^2 k + \sin^2 k - 2\cos k)},$$

and using $1 - \cos k = 2\sin^2 \frac{k}{2}$*, we obtain*

$$|Q|^2 = \frac{1}{1 - 2\sigma(1-\cos k) + 2\sigma^2(1-\cos k)} = \frac{1}{1 + 4\sin^2 \frac{k}{2}\sigma(\sigma - 1)}.$$

Then $|Q|^2 \leq 1$ *provided* $\sigma \geq 1$ *or* $\sigma \leq 0$*. In these cases the scheme is stable. If* $c > 0$*, we need* $\sigma \geq 1$*. Thus the CFL condition for stability is*

$$c \geq \frac{\Delta x}{\Delta t},$$

which is contrary to the usual one for the second-order wave equation.