AEROSPACE ENGINEERING.
Advanced Mathematics. Exam June 17th, 2014.

Select and solve four out of the following five problems (2.5/10 each, 3 hours total time).

1. Suppose you are able to find three linearly independent particular solutions $u_1(x)$, $u_2(x)$ and $u_3(x)$ of the second order linear differential equation $L[y] = f(x)$. What is the general solution?

   Solution. $y(x) = u_1(x) + c_1[u_2(x) - u_1(x)] + c_2[u_3(x) - u_1(x)]$, c_1, c_2 *arbitrary numbers. In fact,* $L[u_2 - u_1] = L[u_2] - L[u_1] = f - f = 0$ *because of the linearity of* L. *Then* $u_2 - u_1$ *and, similarly,* $u_3 - u_1$, *are independent solutions of the homogeneous equation* $L[y] = 0$.

2. Classify the fixed point of the following linear system:
   $\dot{x} = x + y$, $\dot{y} = -x - 2y$,
   and find its general solution.

   Solution. The eigenvalue problem is $\lambda^2 + \lambda - 1 = 0$ *which has solutions* $\frac{1}{2}(-1 \pm \sqrt{5})$ $\lambda_- < 0 < \lambda_+$*) and therefore the origin is a saddle point. The eigenvectors are* $\underline{v}_\pm = (1, \frac{1}{2}(-3 \pm \sqrt{5}))'$. *The general solution is* $\underline{x}(t) = c_-(1, \frac{\sqrt{5}-3}{2})'e^{(-1+\sqrt{5})t/2} + c_+(1, -\frac{\sqrt{5}+3}{2})'e^{-(1+\sqrt{5})t/2}$.

3. A cylindrical slender rod is made out of two metals with positive thermal conductivities, specific heats and mass densities K_1, c_1 and ρ_1 for $0 < x < a$ and K_2, c_2 and ρ_2 for $a < x < L$. Solve the homogeneous heat equation

$$\rho(x)c(x)\frac{\partial u}{\partial t} = K(x)\frac{\partial^2 u}{\partial x^2},$$

   with boundary and initial conditions $u(0,t) = 0$, $u(L,t) = 0$, $u(x,0) = 1$. Recall that the heat flux is continuous throughout the rod.

   Solution. Separating variables, $u = X(x)T(t)$, *we find*

$$\frac{T''}{T} = \frac{K(x)}{\rho(x)c(x)}\frac{X''}{X} = -\lambda.$$

   The eigenvalue problem for X *is*

$$X'' + \lambda\frac{\rho_1 c_1}{K_1}X = 0, \quad 0 < x < a,$$
$$X'' + \lambda\frac{\rho_2 c_2}{K_2}X = 0, \quad a < x < L,$$

with $X(0) = 0$, $X(L) = 0$. The solution of this Sturm-Liouville problem with weight function $w(x) = \rho(x)c(x)/K(x)$ is

$$X(x) = \begin{cases} \sin(r_1 x), & 0 < x < a, \\ \mu \sin[(r_2(L-x)], & a < x < L, \end{cases}$$

where $r_i = \sqrt{\lambda \rho_i c_i / K_i}$, $i = 1, 2$ and μ is a constant. Continuity of X and of the flux $-K_i X'(x)$ at $x = a$ yield

$$\sin(r_1 a) = \mu \sin[r_2(L-a)], \quad K_1 r_1 \cos(r_1 a) = -K_2 r_2 \mu \cos[r_2(L-a)].$$

Elimination of μ produces the eigenvalue equation

$$\tan(r_1 a) = -\frac{K_1 r_1}{K_2 r_2} \tan[r_2(L-a)] \Longrightarrow \tan(r_1 a) = -\sqrt{\frac{K_1 \rho_1 c_1}{K_2 \rho_2 c_2}} \tan[r_2(L-a)].$$

This equation has infinitely many solutions $\lambda_n > 0$, $n = 1, 2, \ldots$ that increase with n. Let $X_n(x)$ be the corresponding eigenfunctions. Their orthogonality condition is

$$\int_0^L w(x) X_n(x) X_m(x) dx = 0, \quad n \neq m; \quad w(x) = \frac{\rho(x)c(x)}{K(x)}.$$

The general solution is $u(x,t) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n t} X_n(x)$. From the initial condition, we obtain

$$1 = \sum_{n=1}^{\infty} a_n X_n(x) \Longrightarrow a_n = \frac{\int_0^L w(x) X_n(x) dx}{\int_0^L w(x) X_n^2(x) dx} \Longrightarrow$$
$$a_n = 4 \frac{\sqrt{\frac{w_1}{\lambda_n}}[1 - \cos(r_1 a)] + \sqrt{\frac{w_2}{\lambda_n}}[\cos(r_2(L-a)) - 1]}{2w_1 a + 2w_2(L-a) - \sqrt{\frac{w_1}{\lambda_n}} \sin(2r_1 a) - \sqrt{\frac{w_2}{\lambda_n}} \sin(2r_2(L-a))}.$$

4. Write a finite difference scheme to solve the telegraph equation:

$$\frac{\partial^2 u}{\partial t^2} + a \frac{\partial u}{\partial t} - b\, u = \frac{\partial^2 u}{\partial x^2},$$

using centered differences to approximate all differentials. Find the CFL condition (ignoring boundary conditions and for a sufficiently small product of the time step times $|a|$).

Solution. $u_j^{n+1} + u_j^{n-1} - 2u_j^n - s\,(u_{j+1}^n + u_{j-1}^n - 2u_j^n) + a\,\Delta t\,(u_j^{n+1} - u_j^{n-1})/2 - b u_j^n (\Delta t)^2 = 0$, *with* $s = (\Delta t/\Delta x)^2$.
The CFL condition is found by substituting $u_j^n = e^{ij\alpha} Q^n$ in the difference equation. We find

$$\left(1 + \frac{a\Delta t}{2}\right) Q^2 - 2\left[1 - s(1 - \cos\alpha) + \frac{b(\Delta t)^2}{2}\right] Q + 1 - \frac{a\Delta t}{2} = 0.$$

The solutions are complex conjugate and have modulus smaller than 1 provided: $0 < a\,\Delta t < 2$ *and*

$$\left[1 - s(1-\cos\alpha) + \frac{b(\Delta t)^2}{2}\right]^2 < \left(1 + \frac{a\Delta t}{2}\right)\left(1 - \frac{a\Delta t}{2}\right) = 1 - \left(\frac{a\Delta t}{2}\right)^2,$$

or, equivalently,

$$-\sqrt{1 - \frac{a^2(\Delta t)^2}{4}} < 1 - 2s\sin^2\frac{\alpha}{2} + b(\Delta t)^2/2 < \sqrt{1 - \frac{a^2(\Delta t)^2}{4}}.$$

This holds if

$$s\sin^2\frac{\alpha}{2} < \frac{1}{2} + \frac{b(\Delta t)^2}{4} + \frac{1}{2}\sqrt{1 - \frac{a^2(\Delta t)^2}{4}} = 1 + \frac{b(\Delta t)^2}{4} - \frac{1}{2}\left(1 - \sqrt{1 - \frac{a^2(\Delta t)^2}{4}}\right),$$

i.e.,

$$s\sin^2\frac{\alpha}{2} < 1 + \frac{b(\Delta t)^2}{4} - \frac{a^2(\Delta t)^2}{8}\frac{1}{1 + \sqrt{1 - \frac{a^2(\Delta t)^2}{4}}}.$$

From this inequality and from $|sin(\alpha/2)| < 1$ *and* $-1 < -1/(1+B)$*, we find the CFL condition*

$$s < 1 - \frac{(\Delta t)^2(a^2 - 2b)}{8} \Leftrightarrow \frac{\Delta x}{\Delta t} > \frac{1}{\sqrt{1 - (a^2 - 2b)\frac{(\Delta t)^2}{8}}}.$$

Here we must have $0 < (a^2 - 2b)\frac{(\Delta t)^2}{8} < 1$*, which places another restriction on the coefficients* a*,* b *and the time step* Δt*.*

5. Solve the following IBVP:

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= c^2\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L,\, t > 0, \\ u(x,0) &= \frac{\partial u}{\partial t}(x,0) = 0, \quad 0 < x < L, \\ u(0,t) &= h(t), \quad u(L,t) = 0, \quad t > 0. \end{aligned}$$

Solution. We try the eigenfunction expansion of the solution, $u(x,t) = \sum_{n=1}^{\infty} a_n(t)\sin\frac{n\pi x}{L}$*. Using*

$$\int_0^L \frac{\partial^2 u}{\partial x^2}\sin\frac{n\pi x}{L}\,dx = -\frac{n\pi}{L}u\cos\frac{n\pi x}{L}\Big|_0^L - \frac{n^2\pi^2}{L^2}\int_0^L u\sin\frac{n\pi x}{L}\,dx,$$

we get

$$\ddot{a}_n + \frac{n^2\pi^2c^2}{L^2}a_n = \frac{2n\pi c^2}{L^2}h(t), \quad a_n(0) = \dot{a}_n(0) = 0.$$

The solution of this IVP can be found by variation of parameters or any other method as

$$a_n(t) = \frac{2c}{L}\int_0^t h(s)\sin\frac{n\pi c(t-s)}{L}\,ds.$$

Then

$$u(x,t) = \frac{2c}{L}\sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}\int_0^t h(s)\sin\frac{n\pi c(t-s)}{L}\,ds.$$

It is also possible to separate, $u(x,t) = h(t)(1-x/L)+v(x,t)$*, and solve the following IBVP with homogeneous b.c.:*

$$\begin{aligned}
&\frac{\partial^2 v}{\partial t^2} - c^2\frac{\partial^2 v}{\partial x^2} = -\Big(1-\frac{x}{L}\Big)\ddot{h}, \quad 0<x<L,\ t>0,\\
&v(x,0) = -\Big(1-\frac{x}{L}\Big)h(0),\ \frac{\partial v}{\partial t}(x,0) = -\Big(1-\frac{x}{L}\Big)\dot{h}(0), \quad 0<x<L,\\
&v(0,t) = 0, \quad v(L,t) = 0, \quad t>0.
\end{aligned}$$

By eigenfunction expansion, we find

$$\ddot{b}_n + \frac{n^2\pi^2c^2}{L^2}b_n = -\frac{2}{n\pi}\ddot{h}(t), \quad b_n(0) = -\frac{2}{n\pi}h(0),\ \dot{b}_n(0) = -\frac{2}{n\pi}\dot{h}(0),$$

for the Fourier coefficients of $v(x,t)$*. Then* $v(x,t) = \sum_{n=1}^{\infty} b_n(t)\sin\frac{n\pi x}{L}$*, with*

$$b_n(t) = -\frac{4c}{n\pi L}\int_0^t \ddot{h}(s)\sin\frac{n\pi c(t-s)}{L}\,ds - \frac{2}{n\pi}h(0)\cos\frac{n\pi ct}{L} - \frac{2L}{n^2\pi^2c}\dot{h}(0)\sin\frac{n\pi ct}{L}.$$

Yet another way to solve this problem is to use the method of characteristics taking into account the infinitely many reflections at the boundaries.