AEROSPACE ENGINEERING.
Advanced Mathematics. Final exam. January 23rd, 2013.

Select and solve four out of the following five problems.

1. A nanotube resonator may be modeled as a thin cylinder of several nanometers radius supported on two contacts. The natural transversal vibrations of a nanotube with simply supported ends satisfy the equation: $\partial_t^2 u = -a^2 \partial_x^4 u$, with boundary conditions $u(0,t) = \partial_x^2 u(0,t) = 0$, $u(L,t) = \partial_x^2 u(L,t) = 0$. Assume that the initial conditions are $u(x,0) = f(x)$ and $\partial_t u(x,0) = 0$. Solve this initial-boundary value problem specifying the natural frequencies of the nanotube. *Hint: Separate variables and use that the quartic roots of 1 are* ± 1, $\pm i$.

   Solution. Separating variables, $u(x,t) = X(x)T(t)$, *we have* $\frac{T''}{a^2 T} = -\frac{X^{(4)}}{X} = -\lambda$. *Choosing* $\lambda > 0$, *the spatial part of the solution yields the following eigenvalue problem*

$$X^{(4)} = \lambda X, \quad X(0) = X''(0) = 0, \quad X(L) = X''(L) = 0,$$

   whose solutions are $X(x) = a_1 \cosh(\lambda^{1/4}x) + a_2 \sinh(\lambda^{1/4}x) + b_1 \cos(\lambda^{1/4}x) + b_2 \sin(\lambda^{1/4}x)$. *(Trying* $X = e^{qx}$, *we get* $q^4 = \lambda$ *whose solutions are* $q/\lambda^{1/4} = \pm 1, \pm i$. *These roots provide the indicated general solution.) The boundary conditions at* $x = 0$ *produce* $a_1 + b_1 = 0$ *and* $\sqrt{\lambda}(a_1 - b_1) = 0$. *This gives* $a_1 = b_1 = 0$. *Then the boundary conditions at* $x = L$ *yield* $a_2 \sinh(\lambda^{1/4}L) + b_2 \sin(\lambda^{1/4}L) = 0$ *and* $\sqrt{\lambda}[a_2 \sinh(\lambda^{1/4}L) - b_2 \sin(\lambda^{1/4}L)] = 0$. *This gives* $a_2 = 0$ *and* $b_2 \sin(\lambda^{1/4}L) = 0$. *Then* $\lambda^{1/4}L = n\pi$, *where* $n = 1, 2, \ldots$. *Thus the eigenvalues are* $\lambda_n = (n\pi/L)^4$, $n = 1, 2, \ldots$ *with eigenfunctions* $X_n(x) = \sin \frac{n\pi x}{L}$. *If* $\lambda \leq 0$, *then* $X(x) = 0$. *The equation for* $T(t)$ *is* $T'' + T(n^2\pi^2 a/L^2)^2 = 0$ *whose general solution is* $T = A_n \cos \frac{n^2\pi^2 at}{L^2} + B_n \cos \frac{n^2\pi^2 at}{L^2}$. *The natural frequencies of the nanotube are therefore* $\omega_n = \frac{n^2\pi^2 a}{L^2}$, *with* $n = 1, 2, \ldots$. *The superposition principle yields*

$$u(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos \frac{n^2\pi^2 at}{L^2} + B_n \cos \frac{n^2\pi^2 at}{L^2} \right) \sin \frac{n\pi x}{L}.$$

   The initial condition $\partial_t u(x,0) = 0$ *gives* $B_n = 0$ *and the other initial condition produces*

$$f(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L}.$$

Then

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx,$$

and

$$u(x,t) = \sum_{n=1}^{\infty} A_n \cos\frac{n^2\pi^2 at}{L^2}\,\sin\frac{n\pi x}{L}.$$

2. The truncation error of the explicit finite difference scheme for the homogeneous heat equation $\partial_t u = k\partial_x^2 u$ is $\tau_j^{(m)}(\Delta t,\Delta x) = \frac{u_j^{(m+1)}-u_j^{(m)}}{\Delta t} - \frac{k}{(\Delta x)^2}(u_{j+1}^{(m)}+u_{j-1}^{(m)}-2u_j^{(m)})$, where $u_j^{(m)}$ should be replaced by $u(j(\Delta x),m(\Delta t))$.

   (a) Calculate $|\tau_j^{(m)}|$. For which value of $s = k(\Delta t)/(\Delta x)^2$ is this error smallest?

   (b) Calculate $|\tau_j^{(m)}|$ for the specific value of s found in (a).

   Solution. (a) Taylor expand $u_j^{(m+1)}$ and $u_{j\pm1}^{(m)}$ up to terms of order $(\Delta t)^2$ and $(\Delta x)^4$. Then use the heat equation to get rid of the time derivatives in the result:

$$\tau_j^{(m)} = \frac{\Delta t}{2}\partial_t^2 u_j^{(m)} + \frac{(\Delta t)^2}{2}\partial_t^3 u_j^{(m)} - \frac{k(\Delta x)^2}{12}\partial_x^2 u_j^{(m)} - \frac{2k(\Delta x)^4}{6!}\partial_x^6 u_j^{(m)} + \dots$$

$$\tau_j^{(m)} = \frac{k(\Delta x)^2}{2}\frac{k(\Delta x)}{2}\left(\frac{k(\Delta t)}{(\Delta x)^2}-\frac{1}{6}\right)\partial_x^4 u_j^{(m)} + \frac{k(\Delta x)^4}{6}\left(\frac{k^2(\Delta t)^2}{(\Delta x)^4}-\frac{1}{60}\right)\partial_x^6 u_j^{(m)} + \dots$$

   Clearly the error $|\tau_j^{(m)}|$ is smallest for $s = 1/6$.
   (b) Inserting $s = 1/6$ in the last formula for the error, we get $\tau_j^{(m)} = \frac{k(\Delta x)^4}{540}\partial_x^6 u_j^{(m)}$ plus higher order terms.

3. Solve

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}, \quad t>0,$$

   with initial conditions $u(x,0) = 0$, $\frac{\partial u}{\partial t}(x,0) = 0$, $c > 0$, subject to $u(x,t) = h(t)$ along $x = \frac{ct}{2}$.

   Draw the plane (x,t) with the straight lines $x = \pm ct$ and $x = ct/2$. There are four regions to consider for $t > 0$. In regions $x < -ct$ and $x > ct$ the initial conditions imply that $u(x,t) = 0$. In region $\frac{ct}{2} < x < ct$, the characteristics with positive slope passing through point (x,t) is $c(\tau - t) = \xi - x$. This line intersects $\xi = c\tau/2$ at $\xi = \frac{c\tau}{2} = ct - x$. Then the parallelogram rule implies $u(x,t) = h(2t - 2x/c)$ ($u = 0$ for the two vertices on $x = ct$). In the region $-ct < x < \frac{ct}{2}$, the characteristics with negative slope passing through point (x,t) is $c(\tau - t) = -(\xi - x)$. This line intersects $\xi = c\tau/2$ at $\xi = \frac{c\tau}{2} = \frac{x+ct}{3}$. Then the parallelogram rule implies $u(x,t) = h(\frac{2}{3c}(x+ct))$ ($u = 0$ for the two vertices on $x = -ct$).
   Another solution method. In regions $x < -ct$ and $x > ct$ the initial conditions imply that $u(x,t) = 0$. If $u(x,t) = F(x-ct) + G(x+ct)$, then $F(\xi) = G(\xi) = 0$ for $\xi > 0$ (region $x > ct$) and $F(\xi) = G(\xi) = 0$ for $\xi < 0$ (region $x < -ct$). In the region

$\frac{ct}{2} < x < ct$ *which is separated from* $x > ct$ *by the line* $x = ct$, $x - ct < 0$ *but* $x + ct > 0$ *so that* $G(x + ct) = 0$ *and* $u = F(x - ct)$ *with* $-\frac{ct}{2} < (x - ct) < 0$. *Then* $h(t) = F(\frac{ct}{2} - ct) = F(-\frac{ct}{2})$ *by the boundary condition. This implies* $F(\xi) = h(-2\xi/c)$ *for* $-\frac{ct}{2} < \xi < 0$. *Thus* $u(x,t) = h(2t - 2x/c)$ *in the region* $\frac{ct}{2} < x < ct$. *In the region* $-ct < x < \frac{ct}{2}$, $x - ct < 0$, $0 < (x + ct) < 3c/2$, *so that* $F(x - ct) = 0$ *and therefore* $u = G(x + ct)$. *At the boundary,* $h(t) = G(\frac{ct}{2} + ct) = G(\frac{3ct}{2})$. *Then* $G(\xi) = h(\frac{2\xi}{3c})$ *and* $u(x,t) = h(\frac{2(x+ct)}{3c})$.

4. The two ends of a cylindrical metal bar of length L are kept at zero temperature and its lateral surface exchanges heat with the outside (at fixed temperature u_0) according to Newton's cooling law. Initially the bar temperature is zero. Find the temperature at each point of the bar. *Hint. Use* $\partial_t u = k\partial_x^2 u - h(u - u_0)$.

   Solution. We solve the temperature equation with Dirichlet boundary conditions and zero initial condition: $u = u_0 + v(x) + w(x,t)$, *where*

$$kv'' - hv = 0, \quad v(0) = -u_0, \quad v(L) = -u_0.$$

   We get

$$v(x) = -u_0 \frac{\sinh\left[(L-x)\sqrt{\frac{h}{k}}\right] + \sinh x\sqrt{\frac{h}{k}}}{\sinh L\sqrt{\frac{h}{k}}}.$$

   Then $w(x,t)$ *satisfies the homogeneous equation with zero-data Dirichlet boundary conditions* $\partial_t w = k\partial_x^2 w - hw$, $w(0,t) = 0 = w(L,t)$, *and initial conditions* $w(x,0) = -u_0$. *Inserting the eigenfunction expansion:* $w(x,t) = \sum_{n=1}^{\infty} b_n(t) \sin\frac{n\pi x}{L}$, *we obtain the ODE* $b_n' = -(h + kn^2\pi^2/L^2)b_n$ *to be solved with initial condition* $b_n(0) = -u_0\frac{2}{L}\int_0^L \sin\frac{n\pi x}{L}dx = \frac{2u_0}{n\pi}[(-1)^n - 1]$. *The result is* $u = u_0 + v(x) + w(x,t)$, *with*

$$w(x,t) = -\frac{4hL^2u_0}{\pi k}\sum_{j=1}^{\infty} \frac{\sin\frac{(2j-1)\pi x}{L}}{(2j-1)[(2j-1)^2\pi^2 k + hL^2]} e^{-[h+(2j-1)^2\pi^2 k/L^2]t}.$$

   Another method. Eigenfunction expansion $u = \sum_{n=1}^{\infty} b_n(t)\sin\frac{n\pi x}{L}$, *the boundary and initial conditions yield*

$$b_n' + \left(k\frac{n^2\pi^2}{L^2} + h\right)b_n = \frac{2}{n\pi}[1 - (-1)^n]hu_0, \quad b_n(0) = 0.$$

   The solution is $b_n(t) = \frac{2}{n\pi}\frac{1-(-1)^n}{h+kn^2\pi^2/L^2}hu_0[1 - e^{-(h+kn^2\pi^2/L^2)t}]$, *thereby producing the equivalent form of the temperature*

$$u(x,t) = \frac{4hL^2u_0}{\pi}\sum_{j=1}^{\infty} \frac{\sin\frac{(2j-1)\pi x}{L}}{(2j-1)[hL^2 + k(2j-1)^2\pi^2]}[1 - e^{-[h+k(2j-1)^2\pi^2/L^2]t}].$$

   This can be checked by Fourier-expanding u_0 *and* $v(x)$ *above.*

5. Solve the Poisson equation:

$$\nabla^2 u = \sin(2x)\cos y,$$

with the boundary conditions

$$u(0,y) = 0, \quad u(x,0) = f(x),$$
$$u(\pi,y) = 0, \quad u(x,L) = 0.$$

Solution. The boundary conditions at $x = 0$ and π are homogeneous and therefore we can use an eigenfunction expansion $u(x,y) = \sum_{n=1}^{\infty} b_n(y)\sin(nx)$. Inserting this in the Poisson equation we find

$$\sum_{n=1}^{\infty}[b_n''(y) - n^2 b_n(y)]\sin(nx) = \sin(2x)\cos y.$$

From this equation we obtain $b_n'' - n^2 b_n = \delta_{n2}\cos y$. The boundary conditions are $b_n(L) = 0$ and $b_n(0) = f_n$, where $f_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx$. For $n \neq 2$, $b_n(y) = \beta_n \sinh[n(y-L)]$ with $\beta_n = -f_n/\sinh(nL)$. For $n = 2$, we have $b_2'' - 4b_2 = \cos y$. This ODE has the particular solution $b_2^p(y) = -\frac{1}{5}\cos y$. Adding a solution of the homogeneous ODE, we obtain $b_2 = a\cosh[2(y-L)] + b\sinh[2(y-L)] - \frac{1}{5}\cos y$. The boundary condition at $y = L$ gives $a - 1/5\cos L = 0$ hence $a = 1/5\cos L$. The boundary condition at $y = 0$ gives $f_2 = \frac{1}{5}[\cos L\cosh(2L) - 1] - b\sinh(2L)$. Then

$$b_2(y) = \frac{\frac{1}{5}[\cos L\cosh(2L) - 1] - f_2}{\sinh(2L)}\sinh[2(y-L)] + \frac{\cosh[2(y-L)] - \cos y}{5}.$$

We find

$$u(x,y) = \left(\frac{\frac{1}{5}[\cos L\cosh(2L) - 1] - f_2}{\sinh(2L)}\sinh[2(y-L)] + \frac{\cosh[2(y-L)] - \cos y}{5}\right)\sin(2x)$$
$$-\frac{f_1}{\sinh L}\sin x\sinh(y-L) - \sum_{n=3}^{\infty} f_n \frac{\sinh[n(y-L)]}{\sinh(nL)}\sin(nx).$$