AEROSPACE ENGINEERING.
Advanced Mathematics. Final exam 06/25/2013, solutions.

Select and solve four out of the following five problems.

1. Answer each of the following questions about the initial value problem (IVP):
$y'' - 4y' + 29y = 0$; $y(0) = -3$, $y'(0) = 4$.

   (a) Convert the IVP into a system of differential equations and write your answer in matrix form.

   (b) Solve the system and use this solution to find the solution to the original 2nd order IVP.

   (c) Sketch the phase portrait for the system and determine the stability of the equilibrium solution.

Solution. First convert to a system in matrix form: $y' = z$, $z' = -29y + 4z$ *which gives*

$$\underline{x}' = \begin{pmatrix} 0 & 1 \\ -29 & 4 \end{pmatrix} \underline{x}, \quad \underline{x}(0) = \begin{pmatrix} -3 \\ 4 \end{pmatrix}.$$

The eigenvalues and eigenvectors for the coefficient matrix are

$$\lambda = 2 + 5i, \quad \underline{\psi} = \begin{pmatrix} 1 \\ 2+5i \end{pmatrix},$$

and their complex conjugates. Then

$$\begin{aligned} e^{(2+5i)t} \begin{pmatrix} 1 \\ -2+5i \end{pmatrix} &= e^{2t}(\cos 5t + i \sin 5t) \begin{pmatrix} 1 \\ -2+5i \end{pmatrix} \\ &= e^{2t} \begin{pmatrix} \cos 5t \\ 2\cos 5t - 5 \sin 5t \end{pmatrix} + ie^{2t} \begin{pmatrix} \sin 5t \\ 5\cos 5t + 2 \sin 5t \end{pmatrix}, \end{aligned}$$

and the general solution of the system is

$$\underline{x} = c_1 e^{2t} \begin{pmatrix} \cos 5t \\ 2\cos 5t - 5 \sin 5t \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} \sin 5t \\ 5\cos 5t + 2 \sin 5t \end{pmatrix}.$$

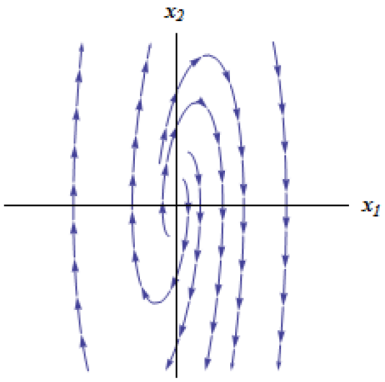
Using the initial condition we find the system of equations

$$c_1 = -3, \quad 2c_1 + 5c_2 = 4 \Longrightarrow c_2 = 2.$$

The solution is then

$$\underline{x}(t) = -3e^{2t} \begin{pmatrix} \cos 5t \\ 2\cos 5t - 5 \sin 5t \end{pmatrix} + 2e^{2t} \begin{pmatrix} \sin 5t \\ 5\cos 5t + 2 \sin 5t \end{pmatrix} = e^{2t} \begin{pmatrix} 2\sin 5t - 3\cos 5t \\ 4\cos 5t + 19 \sin 5t \end{pmatrix}.$$

The solution to the original differential equation is the top row, $y(t) = e^{2t}(-3\cos 5t + 2\sin 5t)$. The velocity vector $\underline{x}'$ corresponding to $(1,0)$ is obtained from the system as $\underline{x}' = (0,-29)$ and it points vertically downwards. We can see that trajectories will cross the x_1 axis going downwards. A sketch of the phase portrait is below. The equilibrium solution in this case is an unstable spiral.



2. A population of fish in a lake grows at a rate proportional to its population. There are originally 600 fish in the lake and in the absence of any outside factors the population will double in 4 months time (assume 4 weeks/month). Each week there is a net migration of 30 fish into the lake and predators eat 47 fish. Derive an ODE for the fish population growth per month.

   (a) Will the fish population survive? If not when do they die out?

   (b) Two months after situation above starts something causes the population of the predators to increase and they will now eat 70 fish/week. The migration of fish into the lake does not change. Does the fish population survive this? If not when do they die out?

Solution. (a) The time unit is one month. Here are the IVPs for this problem:

$$\dot{P} = rP, \quad P(0) = 600,\ P(4) = 1200,$$
$$\dot{P}_1 = rP_1 + 30 \times 4 - 47 \times 4 = rP_1 - 68, \quad P_1(0) = 600.$$

The solution to the first one (so we can find r) is,

$$P(t) = 600e^{rt}, \quad 1200 = 600e^{4r} \Longrightarrow r = \frac{1}{4}\ln 2.$$

The solution to the second IVP is,

$$P_1(t) = \frac{272}{\ln 2} + 207.5869e^{t\ln 2/4}.$$

Both terms in this solution are positive and so the solution will never be zero. This in turn means that the population of fish will survive. (b) After 2 months, $P_1(2) = 685.9853$. The new equation for the fish is

$$\dot{P}_2 = rP_2 + (30 - 70) \times 4 = rP_2 - 160, \quad P_2(2) = P_1(2) = 685.9853.$$

The solution now is

$$P_2(t) = \frac{640}{\ln 2} - \left(\frac{640}{\ln 2} - 685.9853\right) e^{(t-2)\ln 2/4} = \frac{640}{\ln 2} - 237.3395\, e^{(t-2)\ln 2/4}.$$

Clearly the population will not survive and $P_2(t_f) = 0$ at $t_f - 2 \approx 7.84$. Thus the fish die out after 9.84 months (9 months and 25 days).

3. Solve the inhomogeneous heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - u + \sin t, \quad 0 < x < 1,$$

with boundary and initial conditions $u(0,t) = 0$, $u(1,t) = 0$, $u(x,0) = 0$.

Solution. Firstly, find a x-independent solution: $\dot{U} + U = \sin t$, namely, $U(t) = \frac{1}{2}(\sin t - \cos t)$. Rewrite the equation using $u(x,t) = U(t) + e^{-t}v(x,t)$. We find

$$v_t = v_{xx}, \quad v(0,t) = -\frac{e^t}{2}(\sin t - \cos t) = v(1,t), \quad v(x,0) = \frac{1}{2}.$$

As solution we try a Fourier sine series $v(x,t) = \sum_{n=1}^{\infty} b_n(t)\sin(n\pi x)$, thereby obtaining the equation

$$\dot{b}_n + n^2\pi^2 b_n = -\,2n\pi v(x,t)\cos(n\pi x)|_{x=0}^{1} = [1-(-1)^n]n\pi e^t(\cos t - \sin t).$$

The solution is

$$b_n(t) = \frac{[1-(-1)^n]n\pi e^t}{1+(1+n^2\pi^2)^2}[(2+n^2\pi^2)\cos t - n^2\pi^2 \sin t] + ce^{-n^2\pi^2 t}.$$

The initial condition yields

$$\frac{1-(-1)^n}{n\pi} = c + \frac{[1-(-1)^n]n\pi}{1+(1+n^2\pi^2)^2}(2+n^2\pi^2) \Longrightarrow c = \frac{1-(-1)^n}{n\pi}\frac{1+(1+\pi^2n^2)^2 - n^2\pi^2(2+n^2\pi^2)}{1+(1+n^2\pi^2)^2},$$

$$c = \frac{2[1-(-1)^n]}{n\pi[1+(1+n^2\pi^2)^2]}.$$

The overall solution is

$$u(x,t) = \frac{1}{2}(\sin t - \cos t) + \sum_{m=1}^{\infty} \frac{2}{1+(1+(2m-1)^2\pi^2)^2}\Big\{\frac{2e^{-t-(2m-1)^2\pi^2 t}}{(2m-1)\pi}$$
$$+n\pi[(2+(2m-1)^2\pi^2)\cos t - (2m-1)^2\pi^2 \sin t]\Big\}\sin[(2m-1)\pi x].$$

An equivalent solution is obtained by direct eigenfunction expansion of u: $u(x,t) = \sum_{n=1}^{\infty} B_n(t)\sin(n\pi x)$. We find

$$\dot{B}_n + (1+n^2\pi^2)B_n = 2\sin t \int_0^1 \sin(n\pi x)dx = \frac{2}{n\pi}[1-(-1)^n]\sin t,$$

which has to be solved with i.c. $B_n(0) = 0$. We find

$$B_n(t) = \frac{2}{n\pi}\frac{1-(-1)^n}{1+(1+n^2\pi^2)^2}[(1+n^2\pi^2)\sin t - \cos t] + C\,e^{-(1+n^2\pi^2)t},$$

and the i.c. yields

$$C = \frac{2}{n\pi}\frac{1-(-1)^n}{1+(1+n^2\pi^2)^2}.$$

Then

$$u(x,t) = \frac{4}{\pi}\sum_{m=1}^{\infty}\frac{(1+n^2\pi^2)\sin t - \cos t + e^{-(1+n^2\pi^2)t}}{(2m-1)[1+(1+(2m-1)^2\pi^2)^2]}\sin[(2m-1)\pi x].$$

4. A homogeneous rod of length $2L$ is compressed by forces applied on its ends so that it shrinks to a length $2L(1-\epsilon)$. At time $t=0$ the compression forces disappear. Find the subsequent longitudinal displacement of the rod for times $t>0$ provided the middle of the bar is located at $x=0$ and c is the longitudinal sound speed.

   Solution. The displacement of the rod is $u(x,0) = -\epsilon x$ ($[2L(1-\epsilon)-2L]/(2L) = -\epsilon$ is the uniform gradient $\partial u/\partial x$ before $t=0$, which gives $u(x,0) = -\epsilon x$). Solving the homogeneous wave equation with Neumann b.c. at $x = \pm L$ (the applied force at the rod ends is now zero), zero initial velocity and the initial profile $u(x,0) = -\epsilon x$ produces the displacement

$$u(x,t) = \frac{8\epsilon L}{\pi^2}\sum_{n=1}^{\infty}\frac{(-)^{n+1}}{(2n+1)^2}\sin\frac{(2n+1)\pi x}{2L}\,\frac{(2n+1)\pi ct}{2L},$$

   where c is the longitudinal sound velocity.

5. Solve the following Poisson equation inside the half-circle of radius 1 ($0<\theta<\pi$, $0\le r\le 1$):

$$\nabla^2 u = r\sin(2\theta),$$

   with the boundary conditions

$$u(1,\theta) = 0, \quad 0<\theta<\pi,$$
$$\frac{\partial u}{\partial\theta}(r,\theta) = 0, \quad \text{for } \theta = 0 \text{ and for } \theta = \pi.$$

   Solution. We try a particular solution $U = (Ar^3+Br^2)\sin(2\theta)$. Then $(6A+3A-4A)r+2B+B-4B = r$ which gives $A = 1/5$ and leaves B unspecified. Using $B=-A$, $U(1,\theta)=0$. Thus $v = u-U$ solves the Laplace equation with boundary conditions $v(1,\theta)=0$ and $\frac{\partial v}{\partial\theta} = 2r^2(r-1)/5$ for $\theta = 0,\pi$. Using separation of variables, $v(r,\theta) = a_0(r) + \sum_{n=1}^{\infty} a_n(r)\cos(n\theta)$. We now multiply the Laplace equation by $\cos(n\theta)$ and integrate by parts the result, thereby obtaining

$$\frac{\pi}{2}\left(a_n'' + \frac{1}{r}a_n' - \frac{n^2}{r^2}a_n\right) = -\frac{1}{r^2}\frac{\partial v}{\partial\theta}\cos(n\theta)\Big|_0^{\pi} = [1-(-1)^n]\frac{2}{5}(1-r),$$

   for $n\neq 0$ and $a_0'' + a_0'/r = 0$. From these equations, only the odd numbered modes are not zero. For odd n, we have

$$a_n'' + \frac{1}{r}a_n' - \frac{n^2}{r^2}a_n = \frac{8}{5\pi}(1-r), \quad a_n(1) = 0,\ |a_n(0)| < \infty.$$

We obtain

$$a_n(r) = -\frac{8(r^3 - r^n)}{5\pi(9 - n^2)} + \frac{8(r^2 - r^n)}{5\pi(4 - n^2)}, \quad n \neq 3,$$

and

$$a_3(r) = -\frac{8r^3 \ln r}{30\pi} - \frac{8(r^2 - r^3)}{25\pi}.$$

We have found the solution

$$\begin{aligned} u(r,\theta) &= \frac{r^3 - r^2}{5}\sin(2\theta) - \left(\frac{4r^3 \ln r}{15\pi} + \frac{8(r^2 - r^3)}{25\pi}\right)\cos(3\theta) \\ &+ \frac{8}{5\pi}\sum_{m=1, m\neq 2}^{\infty}\left[\frac{r^{2m-1} - r^3}{9 - (2m-1)^2} - \frac{r^{2m-1} - r^2}{4 - (2m-1)^2}\right]\cos[(2m-1)\theta]. \end{aligned}$$

A direct eigenfunction expansion $u(t,\theta) = \frac{1}{2}A_0(r) + \sum_{n=1}^{\infty} A_n(r)\cos(n\theta)$ *yields the following equation for* $a_n(r)$*:*

$$A_n'' + \frac{1}{r}A_n' - \frac{n^2}{r^2}A_n = \frac{2r}{\pi}\int_0^{\pi}\sin(2\theta)\cos(n\theta)d\theta = \frac{r}{\pi}\int_0^{\pi}\{\sin[(2+n)\theta] + \sin[(2-n)\theta]\}d\theta.$$

$$A_n'' + \frac{1}{r}A_n' - \frac{n^2}{r^2}A_n = \frac{r}{\pi}[1 - (-1)^n]\left(\frac{1}{2+n} + \frac{1}{2-n}\right) = \frac{4r}{\pi}\frac{1 - (-1)^n}{4 - n^2}.$$

The right hand side of this equation is 0 for $n = 2$*. Only the odd-numbered modes are not zero. For* $n \neq 3$ *there is a particular solution of the form* $b_n r^3$ *whereas for* $n = 3$ *the particular solution is* $b_3 r^3 \ln r$*. For a right hand side* $\mu_n r$*, we find* $b_n = \mu_n/(9 - n^2)$ *and* $b_3 = \frac{\mu_3}{6}$*. To these solutions we have to add solutions of the homogeneous equation so as to fulfill the boundary conditions* $|A_n(0)| < \infty$ *and* $A_n(1) = 0$*. The result is*

$$u(r,\theta) = -\frac{4}{15\pi}r^3 \ln r\cos(3\theta) + \frac{8}{\pi}\sum_{m=1, m\neq 2}^{\infty}\frac{r^3 - r^{2m-1}}{[4 - (2m-1)^2][9 - (2m-1)^2]}\cos[(2m-1)\theta].$$