AEROSPACE ENGINEERING.
Advanced Mathematics. Final exam. January 16th, 2012.

1. Solve the inhomogeneous heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + e^{-t}, \quad 0 < x < \pi,$$

with boundary and initial conditions $u(0,t) = 0$, $u(\pi,t) = 1$, $u(x,0) = 0$.

Solution. Let us split the solution as $u = \frac{x}{\pi} + v(x,t)$. *$v$ satisfies the same heat equation with homogeneous boundary conditions* $v(0,t) = 0 = v(\pi,t)$ *and initial condition* $v(x,0) = -\frac{x}{\pi}$. *Inserting the eigenfunction expansion,* $v(x,t) = \sum_{n=1}^{\infty} b_n(t) \sin nx$, *into the heat equation for v, we obtain*

$$\frac{db_n}{dt} + n^2 b_n = \frac{2}{\pi}\int_0^{\pi} e^{-t} \sin nx\, dx = \frac{2}{n\pi} e^{-t}[1-(-1)^n].$$

This is equivalent to

$$\frac{d}{dt}\left(e^{n^2 t} b_n\right) = \frac{2}{n\pi} e^{(n^2-1)t}[1-(-1)^n] \Longrightarrow b_n(t) = k_n e^{-n^2 t} + \frac{4e^{-t}}{\pi} \begin{cases} \frac{1}{n(n^2-1)}, & \text{odd } n \neq 1, \\ t, & n = 1, \\ 0, & \text{even } n. \end{cases}$$

Now, $b_n(0) = -\frac{2}{\pi}\int_0^{\pi} \frac{x}{\pi} \sin nx\, dx = (-1)^n \frac{2}{\pi n}$, *which yields*

$$k_n = (-1)^n \frac{2}{\pi n} - \frac{4}{\pi} \begin{cases} \frac{1}{n(n^2-1)}, & \textit{odd } n \neq 1, \\ 0, & \textit{else.} \end{cases}$$

We thus find

$$u(x,t) = \frac{x}{\pi} + \sum_{n=1}^{\infty} (-1)^n \frac{2e^{-n^2 t}}{\pi n} \sin nx + \frac{4te^{-t}}{\pi} \sin x + \frac{1}{\pi} \sum_{m=1}^{\infty} \frac{e^{-t} - e^{-(2m+1)^2 t}}{(2m+1)m(m+1)} \sin[(2m+1)x].$$

2. Consider the first-order wave equation $\partial u/\partial t + c\partial u/\partial x = 0$.

   (a) Discretize this equation by using a centered difference in time and a centered difference in space. What is the truncation error of the resulting scheme?

   (b) Analyze the stability of the resulting scheme (without boundary conditions).

Solution. (a) $\frac{u_j^{(m+1)}-u_j^{(m-1)}}{2\Delta t}+c\frac{u_{j+1}^{(m)}-u_{j-1}^{(m)}}{2\Delta x}=0$ *gives* $u(m+1)_j = u_j^{(m-1)} - s(u_{j+1}^{(m)} - u_{j-1}^{(m)})$ *with* $s = c\Delta t/\Delta x$. *Inserting* $u(m)_j = u(j\Delta x, m\Delta t)$ *in the previous scheme and Taylor expanding,*

$$\frac{u(j\Delta x,(m+1)\Delta t)-u(j\Delta x,(m-1)\Delta t)}{2\Delta t}+c\frac{u((j+1)\Delta x,m\Delta t)-u((j-1)\Delta x,m\Delta t)}{2\Delta x}$$
$$=\frac{(\Delta t)^2}{6}\frac{\partial^3 u}{\partial t^3}(j\Delta x,(m+\tau)\Delta t)+c\frac{(\Delta x)^2}{6}\frac{\partial^3 u}{\partial x^3}(\Delta x(j+\xi),m\Delta t)$$

(τ and ξ are between -1 and 1), and therefore the truncation error is quadratic, comprising a term of order $(\Delta t)^2$ plus a term of order $(\Delta x)^2$.
(b) Substituting $u_j^{(m)} = Q^m e^{ijk}$ *in the difference equation, we find* $Q = Q^{-1} - s(e^{ik} - e^{-ik}) = Q^{-1} - 2is\sin k$. *This is the second order equation* $Q^2 + 2isQ\sin k - 1 = 0$ *whose solutions are:* $Q_\pm = -is\sin k \pm \sqrt{1-s^2\sin^2 k}$. *For* $s|\sin k| < 1$, *we get complex* $Q_\pm$ *such that* $|Q_\pm|^2 = 1$. *For* $s|\sin k| = 1$, *we also get* $|Q_\pm| = 1$, *whereas for* $s|\sin k| > 1$, *the square root is purely imaginary and* $|Q_-| > 1$. *The two first conditions hold if* $s \le 1$ *and therefore the scheme is stable if* $c \le \frac{\Delta x}{\Delta t}$.

3. A semi-infinite homogeneous string has a point mass M attached to its left end at $x = 0$. The string linear mass density and tension are ρ and T, respectively. Then the wave speed is $c = \sqrt{T/\rho}$. The mass is sustained by a massless spring with stiffness constant k. Then the vertical displacement of the mass M is equal to the vertical displacement of the string at $x = 0$: $y(t) = u(0, t)$. Consider that the string and the mass are initially at rest but that the mass has an initial vertical velocity v_0. The initial velocity profile of the string is zero for $x > 0$.

   (a) Write the equation of motion for $y(t)$ by using Newton's second law considering the force $T\frac{\partial u}{\partial x}(0, t)$ that the string exerts on the mass M.

   (b) In the previous equation, calculate the term that depends on u at $x = 0$ in terms of $y(t)$ by solving the wave equation for $x > 0$ with boundary condition $u(0, t) = y(t)$. Substituting this term in the equation of motion for the mass, obtain an ordinary differential equation for $y(t)$.

   (c) Solve the initial-value problem for $y(t)$ in the cases $\rho T > 4Mk$ and $\rho T < 4Mk$.

Solution. (a) The string exerts a force $T\frac{\partial u}{\partial x}$ *at* $x = 0$ *over the mass and therefore Newton's second law yields*

$$M\ddot{y} = -ky + T\frac{\partial u}{\partial x}(0, t),$$

to be solved with initial conditions $y(0) = 0$ *and* $\dot{y}(0) = v_0$. *Note that we are not considering gravity. A constant term* $-Mg$ *in the right-hand side of the previous expression can be absorbed by redefining* $z = y - Mg/k$, $v = u - Mg/k$, *so that* $z(t)$ *and* $v(x, t)$ *obey the same equations, initial and boundary conditions as our functions* $y(t)$ *and* $u(x, t)$, *respectively.*

(b) To find $\frac{\partial u}{\partial x}(0,t)$, we need to solve the following IBVP for the wave equation:

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, & x > 0,\, t > 0, \\ u(0,t) = y(t), & x = 0, \\ u(x,0) = 0, & \frac{\partial u}{\partial t}(x,0) = 0. \end{cases}$$

For $x > ct$, the trivial initial conditions give the solution $u = 0$. For $x < ct$, the boundary condition $u(0,t) = y(t)$ produces the solution $u(x,t) = y(t - x/c)$. Then $\frac{\partial u}{\partial x} = -(1/c)\dot{y}(t - x/c)$ and $T\frac{\partial u}{\partial x}(0,t) = -\frac{T}{c}\dot{y}$. Inserting this in the equation for $y(t)$, we find

$$M\ddot{y} = -ky - \frac{T}{c}\dot{y} \Longrightarrow \ddot{y} + \frac{T}{Mc}\dot{y} + \frac{k}{M}y = 0.$$

Note that the coupling with the string has yielded a friction force on the mass: the mass motion produces waves that escape along the string thereby taking away energy.
(c) The general solution of the previous ODE is found by inserting $y = e^{\lambda t}$, which gives $\lambda^2 + \frac{T}{Mc}\lambda + \frac{k}{M} = 0$, and therefore

$$\lambda = -\frac{T}{2Mc} \pm \sqrt{\frac{T^2}{4M^2c^2} - \frac{k}{M}} = -\frac{T}{2Mc} \pm \frac{\sqrt{\rho T - 4kM}}{2M},$$

where we have used $T = \rho c^2$.
For $\rho T < 4kM$, the square root is purely imaginary and the general solution of the ODE is

$$y(t) = e^{-Tt/(2Mc)}\left(a\cos\frac{\sqrt{\rho T - 4kM}}{2M}t + b\sin\frac{\sqrt{\rho T - 4kM}}{2M}t\right).$$

The boundary condition $y(0) = 0$ gives $a = 0$ and $\dot{y}(0) = v_0$ produces $v_0 = b\frac{\sqrt{\rho T - 4kM}}{2M}$. Thus for $\rho T < 4kM$,

$$y(t) = \frac{2Mv_0}{\sqrt{\rho T - 4kM}}e^{-Tt/(2Mc)}\sin\frac{\sqrt{\rho T - 4kM}}{2M}t.$$

Similarly, for $\rho T > 4kM$, the general solution of the ODE is

$$y(t) = e^{-Tt/(2Mc)}\left(a\cosh\frac{\sqrt{4kM - \rho T}}{2M}t + b\sinh\frac{\sqrt{4kM - \rho T}}{2M}t\right).$$

The boundary condition $y(0) = 0$ gives $a = 0$ and $\dot{y}(0) = v_0$ produces $v_0 = b\frac{\sqrt{4kM - \rho T}}{2M}$. Thus for $\rho T > 4kM$,

$$y(t) = \frac{2Mv_0}{\sqrt{4kM - \rho T}}e^{-Tt/(2Mc)}\sinh\frac{\sqrt{4kM - \rho T}}{2M}t.$$

4. Solve the Poisson equation $\nabla^2 u = y\cos x$ in $(0,\pi)\times(0,L)$ with boundary conditions

$$\frac{\partial u}{\partial x}(0,y) = 0, \quad u(x,0) = 0,$$
$$\frac{\partial u}{\partial x}(\pi,y) = 0, \quad u(x,L) = f(x).$$

Solution. We use the eigenfunction expansion $u(x,y) = \frac{a_0(y)}{2} + \sum_{n=1}^{\infty} a_n(y)\cos nx$ *which is appropriate for the homogeneous Neumann conditions at* $x = 0,\, \pi$*. Inserting in the Poisson equation and differentiating term by term, we obtain*

$$a_n'' - n^2 a_n = y\delta_{n1},$$

where $\delta_{n1} = 1$ *if* $n = 1$ *and 0 otherwise. The boundary conditions for this ODE are* $a_n(0) = 0$ *and* $a_n(L) = \frac{2}{\pi}\int_0^\pi f(s)\cos ns\, ds$*.*
For $n = 0$*, we get* $a_0(y) = \alpha_0 y$ *with* $\alpha_0 = \frac{2}{L\pi}\int_0^\pi f(s)\, ds$*. For* $n > 1$*, the solution satisfying* $a_n(0) = 0$ *is* $a_n(y) = \beta_n \sinh ny$*. The other boundary condition produces* $\beta_n = \frac{2}{\pi \sinh nL}\int_0^\pi f(s)\cos ns\, ds$*. Lastly, for* $n = 1$*, a particular solution of* $a_1'' - a_1 = y$ *is* $-y$*, so that* $a_1(y) = -y + \alpha_1 \cosh y + \beta_1 \sinh y$*. The boundary conditions yield* $\alpha_1 = 0$ *and* $-L + \beta_1 \sinh L = \frac{2}{\pi}\int_0^\pi f(s)\cos s\, ds$*. Then*

$$\beta_1 = \frac{1}{\sinh L}\left(L + \frac{2}{\pi}\int_0^\pi f(s)\cos s\, ds\right).$$

Putting together these results, we obtain the solution

$$u(x,y) = \frac{y}{\pi L}\int_0^\pi f(s)\, ds + \left[\frac{\sinh y}{\sinh L}\left(L + \frac{2}{\pi}\int_0^\pi f(s)\cos s\, ds\right) - y\right]\cos x + \sum_{n=2}^{\infty}\frac{2}{\pi \sinh nL}\left(\int_0^\pi f(s)\cos ns\, ds\right)\sinh ny\cos nx.$$

The same solution is obtained if we separate $u(x,y) = -y\cos x + v(x,t)$*, check that* $v(x,y)$ *solves the Laplace equation with the same boundary conditions as* u *does and then use the method of 1D eigenfunction expansions to get* $v(x,y)$*.*

Another method. Multiply the Poisson equation by the eigenfunction $\cos nx \sin\frac{m\pi y}{L}$ *and integrate over* x *and* y*. The right hand side gives:*

$$\int_0^\pi \cos nx\cos x dx\int_0^L y\sin\frac{m\pi y}{L}dy = \frac{\pi}{2}\delta_{n1}(-1)^{m+1}\frac{L^2}{m\pi}.$$

The integral over the left hand side is done by parts:

$$\begin{aligned}
\int_0^\pi\int_0^L (u_{xx}+u_{yy})\cos nx\sin\frac{m\pi y}{L}dxdy &= \int_0^L [u_x\cos nx]|_0^\pi \sin\frac{m\pi y}{L}dy \\
&+ \int_0^\pi [u_y \sin\frac{m\pi y}{L}]|_0^L \cos nx dx + n\int_0^\pi\int_0^L u_x \sin nx\sin\frac{m\pi y}{L}dxdy \\
-\frac{m\pi}{L}\int_0^\pi\int_0^L u_y\cos nx\cos\frac{m\pi y}{L}dxdy &= n\int_0^L [u\sin nx]|_0^\pi \sin\frac{m\pi y}{L}dy \\
-\frac{m\pi}{L}\int_0^\pi [u\cos\frac{m\pi y}{L}]|_0^L\cos nx dx - \left(n^2+\frac{m^2\pi^2}{L^2}\right)\int_0^\pi\int_0^L u\cos nx\sin\frac{m\pi y}{L}dxdy & \\
= -\frac{m\pi}{L}(-1)^m\int_0^\pi f(x)\cos nx dx - \left(n^2+\frac{m^2\pi^2}{L^2}\right)\int_0^\pi\int_0^L u\cos nx\sin\frac{m\pi y}{L}dxdy. &
\end{aligned}$$

Here we have denoted $u_x = \partial u/\partial x$, $u_{xx} = \partial^2 u/\partial x^2$, etc. We have found

$$\left(n^2 + \frac{m^2\pi^2}{L^2}\right)\int_0^\pi \int_0^L u \cos nx \sin \frac{m\pi y}{L} dxdy = (-1)^m \left[\frac{L^2}{2m}\delta_{n1} - \frac{m\pi}{L}\int_0^\pi f(x)\cos nx dx\right].$$

This result yields the following eigenfunction expansion of the solution:

$$u(x,y) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} c_{nm}\cos nx \sin\frac{m\pi y}{L},$$
$$c_{n,m} = \frac{4}{\pi L(1+\delta_{n0})}\int_0^\pi\int_0^L u\cos nx\sin\frac{m\pi y}{L}dxdy,$$
$$c_{0m} = \frac{2(-1)^m}{m\pi^2}\int_0^\pi f(x)dx, \quad c_{nm} = \frac{4(-1)^m}{n^2+\frac{m^2\pi^2}{L^2}}\left[\frac{L}{2m\pi}\delta_{n1} - \frac{m}{L^2}\int_0^\pi f(x)\cos nx\,dx\right] (n\neq 0).$$

To check that both results are the same, substitute the Fourier sine series

$$y = -\sum_{m=1}^{\infty}\frac{2L(-1)^m}{m\pi}\sin\frac{m\pi y}{L}, \quad \sinh ny = -\sum_{m=1}^{\infty}\frac{2m\pi(-1)^m}{L^2}\frac{\sinh nL}{n^2+\frac{m^2\pi^2}{L^2}}\sin\frac{m\pi y}{L},$$

in the formula for $u(x,y)$ obtained with the method of 1D eigenfunction expansions.

5. Find the general solution of the inhomogeneous system of ODEs:

$$\dot{x} = 4x - 3y,$$
$$\dot{y} = 8x - 6y - 1.$$

Solution. The trace and determinant of the coefficient matrix are -2 and 0, respectively. Thus their eigenvalues are -2 and 0 and the general solution of the homogeneous system is $\underline{x}(t) = \underline{a} + \underline{b}e^{-2t}$, where $\underline{a}$ and $\underline{b}$ are arbitrary vectors. We can use now the method of variation of parameters:

$$\underline{x}(t) = a(t)\begin{pmatrix}\frac{3}{4}\\ 1\end{pmatrix} + b(t)e^{-2t}\begin{pmatrix}1\\2\end{pmatrix}.$$

Inserting this in the system, we obtain

$$\dot{a}\begin{pmatrix}\frac{3}{4}\\1\end{pmatrix} + \dot{b}e^{-2t}\begin{pmatrix}1\\2\end{pmatrix} = -\begin{pmatrix}0\\1\end{pmatrix}.$$

Substituting $\dot{a} = -\frac{4}{3}e^{-2t}\dot{b}$ in the second equation, we get $(2-4/3)e^{-2t}\dot{b} = -1$, i.e., $\dot{b} = -\frac{3}{2}e^{2t}$. The solution of this ODE is

$$b(t) = \beta - \frac{3}{4}e^{2t}.$$

Then $\dot{a} = -\frac{4}{3}e^{-2t}\dot{b} = 2$, and

$$a(t) = \alpha + 2t.$$

We have found

$$\underline{x}(t) = \begin{pmatrix} \frac{3}{4}\alpha + \frac{3}{2}t \\ \alpha + 2t \end{pmatrix} + \Big(\beta e^{-2t} - \frac{3}{4}\Big)\begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

We can also use the method of undetermined coefficients. Since an appropriate constant vector (the eigenvector corresponding to the eigenvalue 0) is a solution of the homogeneous system, the method of undetermined coefficients suggests the following form for the solution of the inhomogeneous system:

$$\underline{x}(t) = \underline{a} + \underline{b}e^{-2t} + \underline{c}t.$$

Inserting this in the system

$$\Big(\underline{\underline{I}}\frac{d}{dt} - \underline{\underline{A}}\Big)\underline{x} = -\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \underline{\underline{A}} = \begin{pmatrix} 4 & -3 \\ 8 & -6 \end{pmatrix},$$

we find

$$-\underline{\underline{A}}\underline{a} - e^{-2t}\Big(2\underline{\underline{I}} + \underline{\underline{A}}\Big)\underline{b} + \underline{c} - t\underline{\underline{A}}\underline{c} = -\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

From this equation, we get

$$\Big(2\underline{\underline{I}} + \underline{\underline{A}}\Big)\underline{b} = 0, \tag{1}$$

$$\underline{\underline{A}}\underline{c} = 0, \tag{2}$$

$$\underline{\underline{A}}\underline{a} = \underline{c} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{3}$$

Calculating the eigenvectors of $\underline{\underline{A}}$ corresponding to the eigenvalues -2 and 0, we obtain from (1) and (2):

$$\underline{b} = \begin{pmatrix} b_1 \\ 2b_1 \end{pmatrix}, \quad \underline{c} = \begin{pmatrix} \frac{3}{4}c_2 \\ c_2 \end{pmatrix},$$

respectively. (3) can be written as

$$\begin{aligned} 4a_1 - 3a_2 &= \tfrac{3}{4}c_2, \\ 8a_1 - 6a_2 &= c_2 + 1. \end{aligned}$$

We now multiply the first of these equations by 2 and subtract the second one from it, thereby obtaining $3c_2/2 - c_2 - 1 = 0$. Thus $c_2 = 2$. The first equation then yields $a_1 = \frac{3}{4}a_2 + \frac{3}{8}$. Putting everything together, we have found the general solution:

$$\underline{x}(t) = \begin{pmatrix} \frac{3}{4}a_2 + \frac{3}{8} \\ a_2 \end{pmatrix} + 2t\begin{pmatrix} \frac{3}{4} \\ 1 \end{pmatrix} + b_1 e^{-2t}\begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

in which a_2 and b_1 are arbitrary constants. This is the same result as given by the other method if we identify $a_2 = \alpha - 3/2$, $b_1 = \beta$.

Yet another method. Note that

$$2\dot{x} - \dot{y} = 1 \Longrightarrow 2x - y = t + c.$$

Then $y = 2x - t - c$ *which, inserted in the equation for* x*, yields*

$$\dot{x} = 4x - 6x + 3t + 3c = -2x + 3(t + c).$$

A particular solution of this ODE is $x_p = \frac{3t}{2} - \frac{3}{4} + \frac{3c}{2}$*. Then*

$$x(t) = \frac{3t}{2} - \frac{3}{4} + \frac{3c}{2} + de^{-2t},$$

and

$$y(t) = 2x - t - c = 3t - \frac{3}{2} + 3c + 2de^{-2t} - t - c = 2t - \frac{3}{2} + 2c + 2de^{-2t}.$$

This solution of the system coincides with that given by the the method of variation of parameters if $c = \alpha/2$ *and* $d = \beta$*.*