AEROSPACE ENGINEERING.
Advanced Mathematics. Final exam. June 27th, 2012.

1. Consider the general second-order Runge-Kutta scheme to solve the initial-value problem $y' = f(t, y)$ with $y(0) = y_0$:

$$y_{n+1} = y_n + h\left((1 - \frac{1}{2\alpha})f(t_n, y_n) + \frac{1}{2\alpha}f(t_n + \alpha h, y_n + \alpha h f(t_n, y_n))\right),$$

where $0 < \alpha \leq 1$ and $t_{n+1} = t_n + h$ ($\alpha = \frac{1}{2}, \frac{2}{3}, 1$ produce commonly used schemes). Calculate the truncation error of this scheme and show that it is consistent.
Solution. The truncation error is

$$\tau_n(h) = \frac{y(t_{n+1}) - y(t_n)}{h} - \left(1 - \frac{1}{2\alpha}\right)f(t_n, y(t_n)) - \frac{1}{2\alpha}f(t_n + \alpha h, y(t_n) + \alpha h f(t_n, y(t_n))).$$

We now expand the last term in powers of h:

$$f(t_n + \alpha h, y(t_n) + \alpha h f(t_n, y(t_n)) = f + (f_t + f f_y)\alpha h + (f_{tt} + 2f f_{ty} + f_t f_y + f f_y^2 + f_{yy} f^2)\frac{\alpha^2 h^2}{2} + O(h^3).$$

Here $f = f(t_n, y(t_n))$, $f_t = \frac{\partial f}{\partial t}(t_n, y(t_n))$, etc. Using the differential equation, $y' = f(t, y)$, $y'' = f_t + f_y y' = f_t + f f_y$ and $y''' = f_{tt} + 2f f_{ty} + f_t f_y + f f_y^2 + f_{yy} f^2$. Then we have

$$f(t_n + \alpha h, y(t_n) + \alpha h f(t_n, y(t_n)) = y' + y''\alpha h + y'''\frac{\alpha^2 h^2}{2} + O(h^3).$$

Inserting this in the formula for the truncation error and Taylor expanding, we find

$$\tau_n(h) = \frac{h^2 y'''}{2}\left(\frac{1}{3} - \frac{\alpha}{2}\right) + O(h^3).$$

Therefore $\tau_n = O(h^2)$ which tends to zero with h: the scheme is consistent. It is usually second order but it becomes third order if we choose $\alpha = 2/3$.

2. Classify the fixed point of the following linear system for $\beta = 0$:
$\dot{x} = x + y$, $\dot{y} = -x - 2y + \beta \cos t$.
Find the general solution of the previous inhomogeneous system for nonzero β.
Solution. The trace and the determinant of the coefficient matrix are both -1. Then the eigenvalue equation is $\lambda^2 + \lambda - 1 = 0$ with solutions $\lambda_\pm = (-1 \pm \sqrt{5})/2$. One eigenvalue is positive, the other negative and therefore the origin is a saddle point. To find the corresponding eigenvectors, we solve $(1 - \lambda_\pm)a + b = 0$, which gives

$b = (\lambda_\pm - 1)a$. *The phase portrait is obvious once the straight lines* $y = (\lambda_\pm - 1)x$ *are depicted.*

Provided d *is not an eigenvalue, the inhomogeneous equation* $\dot{\underline{x}} = \underline{\underline{A}}\,\underline{x} + \underline{c}e^{dt}$ *has a particular solution* $\underline{\mu}e^{dt}$*, where* $\underline{\mu}$ *is a vector to be determined. We find*

$$(d\underline{\underline{I}} - \underline{\underline{A}})\underline{\mu} = \underline{c} \Longrightarrow \underline{\mu} = (d\underline{\underline{I}} - \underline{\underline{A}})^{-1}\underline{c}.$$

We have

$$d\underline{\underline{I}} - \underline{\underline{A}} = \begin{pmatrix} d-1 & -1 \\ 1 & d+2 \end{pmatrix} \Longrightarrow (d\underline{\underline{I}} - \underline{\underline{A}})^{-1} = \frac{1}{d^2+d-1}\begin{pmatrix} d+2 & 1 \\ -1 & d-1 \end{pmatrix}.$$

Then we obtain for $d = i$*:*

$$(i\underline{\underline{I}} - \underline{\underline{A}})^{-1}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{i-2}\begin{pmatrix} i+2 & 1 \\ -1 & i-1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} -3-4i & -2-i \\ 2+i & 3-i \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} -2-i \\ 3-i \end{pmatrix}.$$

The particular solution is

$$\underline{x}_p(t) = \frac{\beta}{5}Re\begin{pmatrix} -(2+i)e^{it} \\ (3-i)e^{it} \end{pmatrix} = \frac{\beta}{5}\begin{pmatrix} \sin t - 2\cos t \\ 3\cos t + \sin t \end{pmatrix},$$

and the general solution with arbitrary constants $c_\pm$ *is*

$$\underline{x}(t) = c_+ e^{(\sqrt{5}-1)t/2}\begin{pmatrix} 1 \\ \frac{\sqrt{5}-3}{2} \end{pmatrix} + c_- e^{-(\sqrt{5}+1)t/2}\begin{pmatrix} 1 \\ -\frac{\sqrt{5}+3}{2} \end{pmatrix} + \frac{\beta}{5}\begin{pmatrix} \sin t - 2\cos t \\ 3\cos t + \sin t \end{pmatrix}.$$

3. Solve the inhomogeneous heat equation

$$\frac{\partial u}{\partial t} + 5\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2} + e^{-t}, \quad 0 < x < 1,$$

with boundary and initial conditions $u(0,t) = 0$, $u(1,t) = 0$, $u(x,0) = 0$. Hint: Try a substitution $u(x,t) = e^{cx}v(x,t)$ (with an appropriate c) to get rid of the first-order derivative $\partial v/\partial x$. The resulting equation for v is easy to solve with standard techniques (eigenfunction expansions).
Solution. Let $u(x,t) = e^{cx}v(x,t)$*. We find*

$$e^{cx}\frac{\partial v}{\partial t} + 5e^{cx}\frac{\partial v}{\partial x} + 5ce^{cx}v = e^{cx}\frac{\partial^2 v}{\partial x^2} + 2ce^{cx}\frac{\partial v}{\partial x} + c^2 e^{cx}v + e^{-t}.$$

The derivatives $\frac{\partial v}{\partial x}$ *disappear from this equation if we choose* $c = 5/2$*. The result is*

$$\frac{\partial v}{\partial t} + \frac{25}{2}v = \frac{\partial^2 v}{\partial x^2} + \frac{25}{4}v + e^{-t-5x/2} \Longrightarrow \frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} + \frac{25}{4}v = e^{-t-5x/2}.$$

The boundary and initial conditions are now: $v(0,t) = v(1,t) = 0$*,* $v(x,0) = 0$*. We assume now that the solution is an eigenfunction expansion corresponding to the homogeneous Dirichlet problem:*

$$v(x,t) = \sum_{n=1}^{\infty} b_n(t)\sin(n\pi x).$$

This series can be differentiated twice term by term with respect to x because the boundary conditions are homogeneous. Substituting in the PDE, we get

$$\dot{b}_n + \Big(\frac{25}{4} + n^2\pi^2\Big) b_n = 2e^{-t} \int_0^1 e^{-5x/2} \sin(n\pi x)\, dx = 2n\pi \frac{1 - (-1)^n e^{-5/2}}{n^2\pi^2 + \frac{25}{4}}\, e^{-t},$$

to be solved with initial condition $b_n(0) = 0$. Note that

$$2 \int_0^1 e^{-5x/2} \sin(n\pi x)\, dx = 2Im \int_0^1 e^{(in\pi - 5/2)x} dx = 2n\pi \frac{1 - (-1)^n e^{-5/2}}{\frac{25}{4} + n^2\pi^2}.$$

The solution of the initial-value problem for $b_n(t)$ is

$$b_n(t) = \frac{2n\pi[1 - (-1)^n e^{-5/2}]}{\frac{21}{4} + n^2\pi^2} \frac{e^{-t} - e^{-(n^2\pi^2 + 25/4)t}}{\frac{25}{4} + n^2\pi^2}.$$

This yields the following solution of the original initial-value problem

$$u(x,t) = \sum_{n=1}^{\infty} \frac{2n\pi[1 - (-1)^n e^{-5/2}]}{\frac{21}{4} + n^2\pi^2} \frac{e^{-t} - e^{-(n^2\pi^2 + 25/4)t}}{\frac{25}{4} + n^2\pi^2}\, e^{5x/2} \sin(n\pi x).$$

An equivalent form can be obtained by assuming $u(x,t) = -e^{-t} + e^{5x/2} V(x,t)$. The result is

$$u(x,t) = -e^{-t} + \sum_{n=1}^{\infty} \frac{2n\pi[1 - (-1)^n e^{-5/2}]}{\frac{21}{4} + n^2\pi^2} \left(e^{-t} - \frac{e^{-(n^2\pi^2 + 25/4)t}}{\frac{25}{4} + n^2\pi^2}\right) e^{5x/2} \sin(n\pi x).$$

Using now $1 = \sum_{n=1}^{\infty} 2n\pi \frac{1-(-1)^n e^{-5/2}}{\frac{25}{4} + n^2\pi^2} e^{5x/2} \sin(n\pi x)$, we see that these two forms of the solution are the same.

4. Solve the first-order wave equation $\partial u/\partial t + c\, \partial u/\partial x = 0$ with $u = \cos t$ at $x = t$ (i.e. with the condition $u(t,t) = \cos t$) for $c \neq 1$.
Solution. The general solution of the first-order wave equation is $u = f(x - ct)$, where $f(x)$ is arbitrary. This can be checked immediately or derived by the method of characteristics. At $x = t$, we know that $\cos t = u(t,t) = f(t - ct)$. If we call $\xi = t - ct$, $t = \xi/(1 - c)$, and we find $f(\xi) = \cos \frac{\xi}{1-c}$. Replacing $\xi = x - ct$, we get

$$u(x,t) = \cos\Big(\frac{x - ct}{1 - c}\Big).$$

5. Solve the Laplace equation $\nabla^2 u = 0$ in $(0,1) \times (0,L)$ with boundary conditions

$$u(0,y) = 1, \quad \frac{\partial u}{\partial y}(x,0) = 0,$$
$$\frac{\partial u}{\partial x}(1,y) = f(y), \quad \frac{\partial u}{\partial y}(x,L) = 0.$$

Solution. Clearly, $u = 1 + v(x,y)$ solves the Laplace equation with the same boundary conditions except that $v(0,y) = 0$. At $y = 0$ and $y = L$ there are homogeneous

Neumann boundary conditions. Then we should use eigenfunctions associated to the y variable, namely $\cos\frac{n\pi y}{L}$*, with* $n = 0, 1, \ldots$*. Inserting an expansion* $v(x,y) = \frac{1}{2}a_0(x) + \sum_{n=1}^{\infty} a_n(x)\cos\frac{n\pi y}{L}$ *in the Laplace equation, we obtain*

$$a_n'' - \frac{n^2\pi^2}{L^2}\,a_n = 0, \quad a_n(0) = 0, \quad a_n'(1) = \frac{2}{L}\int_0^L f(y)\cos\frac{n\pi y}{L}\,dy.$$

The solution of this boundary-value problem is $a_n(x) = \mu_n \sinh\frac{n\pi x}{L}$ *with* $a_n'(1) = (\mu_n n\pi/L)\cosh\frac{n\pi}{L}$ *which gives* μ_n*. For* $n = 0$*,* $a_0(x) = \mu_0 x$ *with* $\mu_0 = a_0'(1)$*. The solution of the boundary-value problem for the Laplace equation is*

$$u(x,y) = 1 + \frac{x}{L}\int_0^L f(s)ds + \sum_{n=1}^{\infty}\frac{2}{n\pi\cosh(n\pi/L)}\cos\frac{n\pi y}{L}\,\sinh\frac{n\pi x}{L}\int_0^L f(s)\cos\frac{n\pi s}{L}\,ds.$$

We can also use the two-dimensional eigenfunctions of the Laplacian with homogeneous boundary conditions corresponding to those of the problem statement. These eigenfunctions are $\phi_{m,n}(x,y) = \sin[(m-\frac{1}{2})\pi x]\cos\frac{n\pi y}{L}$*, with* $m = 1, 2, \ldots$*,* $n = 0, 1, 2, \ldots$*, and the corresponding eigenvalues are* $-\lambda_{m,n}$ *with* $\lambda_{m,n} = [(m-1/2)^2 + n^2/L^2]\pi^2$*. We multiply now* $\nabla^2 u$ *by* $\phi_{m,n}$*, integrate over the rectangle and use integration by parts to get*

$$0 = \int_0^1\int_0^L \phi_{m,n}\nabla^2 u\,dxdy = \int_0^L (\phi_{m,n}\partial_x u - u\partial_x\phi_{m,n})\Big|_0^1 dy + \int_0^1\int_0^L u\nabla^2\phi_{m,n}dxdy$$
$$\Longrightarrow \lambda_{m,n}\int_0^1\int_0^L u\,\phi_{m,n}dxdy = \int_0^L\left[(-1)^{m+1}f(y) + \frac{2m-1}{2}\pi\right]\cos\frac{n\pi y}{L}dy.$$

The second term is zero unless $n = 0$ *and the left hand side of this expression gives the Fourier coefficient of* $u(x,y)$*. Then we obtain an equivalent form of the solution*

$$u(x,y) = -\sum_{m=1}^{\infty}\left\{\left[\frac{4}{\pi(2m-1)} + \frac{8(-1)^m}{\pi^2(2m-1)^2}\frac{1}{L}\int_0^L f(s)\,ds\right]\sin\frac{(2m-1)\pi x}{2}\right\}$$
$$-\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left\{\frac{2}{\pi^2}\left[\frac{(-1)^m\frac{2}{L}\int_0^L f(s)\cos\frac{n\pi s}{L}ds}{\left(m-\frac{1}{2}\right)^2 + \frac{n^2}{L^2}}\right]\sin\frac{(2m-1)\pi x}{2}\,\cos\frac{n\pi y}{L}\right\}.$$